Projective Geometry Description and Applications for Quantum Entanglement

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Outline

Segre Varieties

Definition (Segre embedding)

The Segre embedding is an injective morphism defined as follows:

$$\Sigma : \mathbb{P}_k^{d_1-1} \times \mathbb{P}_k^{d_2-1} \hookrightarrow \mathbb{P}_k^{d_1 d_2 - 1}, \tag{1}$$

which takes a pair of points $([a],[b]) \in \mathbb{P}_k^{d_1-1} \times \mathbb{P}_k^{d_2-1}$ *to their products:*

$$([a_0 : \cdots : a_{d_1-1}], [b_0 : \cdots : b_{d_2-1}]) \mapsto [a_0 b_0 : a_0 b_1 : \cdots : a_{d_1-1} b_{d_2-1}]. \tag{2}$$

Definition (Segre variety)

The image of the Segre embedding is a variety, called the **Segre variety**.

Veronese Varieties

Definition (Veronese embedding)

The degree n Veronese embedding

$$\Omega : \mathbb{P}_k^d \hookrightarrow \mathbb{P}_k^m \ , \qquad (3)$$

is an injective morphism defined by:

$$[a_0 : \cdots : a_d] \mapsto [f_0(a_0, \cdots, a_n) : \cdots : f_m(a_0, \cdots, a_n)] \ , \qquad (4)$$

where $f_i, \ i = 0 \cdots, m$ *are all of the d-variate degree-n monomials.*

Definition (Veronese variety)

The image of the Veronese embedding is a variety, called the **Veronese variety**.

Grassmann Varieties

Definition (Plucker embedding)

Let V be a finite-dimensional vector space over k and denote the Grassmannian by $\mathbb{G}(d, V)$. The degree d Plucker embedding

$$\mathbb{G}(d, V) \hookrightarrow \mathbb{P}_k^{\binom{\dim V}{d}-1}, \tag{5}$$

is an injective morphism defined by:

$$W \mapsto [w_1 \wedge \cdots \wedge w_d], \quad \text{where } \{w_1, \cdots, w_n\} \text{ is a basis for } W \subseteq V. \tag{6}$$

Definition (Grassmann variety)

The image of the Plucker embedding is a variety, called the **Grassmann variety**.

Secant Variety

Definition (Projective line)

Let $A = [a_0 : \cdots : a_n]$ and $B = [b_0 : \cdots : b_n] \in \mathbb{P}^n$ be two distinct points in $\mathbb{P}^n$, they correspond to two vectors $\mathbf{a} = (a_0, \ldots, a_n)$ and $\mathbf{b} = (b_0, \ldots, b_n) \in k^{n+1}$, which span a plane $\Lambda \subset k^{n+1}$, whose vectors are of the form $u\mathbf{a} + v\mathbf{b}$ with $u, v \in k$.
The corresponding line $\overline{AB} = \mathbb{P}(\Lambda) \subset \mathbb{P}^n$ is:

$$\mathbb{P}(\Lambda) = \{[ua_0 + vb_0 : \cdots : ua_n + vb_n] \mid u, v \in k,\ \text{not both zero}\}\ . \tag{7}$$

Example

The projective line $\overline{pq}$ joining $p = [1 : 0 : 0 : 0]$ and $q = [a : b : c : d]$ has points with coordinates:

$$[u + va : vb : vc : vd]\ . \tag{8}$$

Secant Variety

Definition (Secant variety)

Let projective varieties X and Y be subvarieties of a projective variety. The joining of X and Y is given by the Zariski closure, of the lines from one to the other,

$$\mathbb{J}(X,Y) = \overline{\bigcup_{x\in X, y\in Y} \overline{xy}} \tag{9}$$

where $\overline{xy}$ is the projective line that includes both x and y called a **chord** *or a* **secant line**. *If $X = Y$, then the variety is called a* **secant variety of** *X and denoted by $\sigma(X)$, that is,*

$$\sigma(X) = \mathbb{J}(X,X) = \overline{\bigcup_{x,y\in X} \overline{xy}}\ . \tag{10}$$

Secant Variety

Definition

Let $X \subset \mathbb{P}^N$ be a projective variety. The $k-$secant variety of X, denoted $\sigma_k(X)$, is the Zariski closure of the union of all projective linear spaces spanned by $k+1$ points on X. In other words, $\sigma_k(X)$ consists of all points in $\mathbb{P}^N$ that lie on secant k-planes of X:

$$\sigma_k(X) = \overline{\bigcup_{p_0,\dots,p_k \in X} \langle p_0, \dots, p_k \rangle}\,, \tag{11}$$

where $\langle p_0, \dots, p_k \rangle$ denotes the projective span of the points $p_0, \dots, p_k$.

Secant Variety

Theorem

If the projective variety $X \subseteq \mathbb{P}(k^d)$ is non-degenerate, i.e., it is not contained in a linear subspace of $\mathbb{P}(k^d)$, there is a sequence of inclusions given by:

$$X \subseteq \sigma(X) \subseteq \sigma_3(X) \subseteq \cdots \subseteq \sigma_N(X) = \mathbb{P}(k^d) \ , \tag{12}$$

where N is the smallest integer such that the N^{th} secant variety fills the ambient space.

Definition

The **proper secant variety** *is defined as $\varsigma_k = \sigma_k \backslash \sigma_{k-1}$. We defined $\sigma_2(X) = \sigma(X)$ and $\sigma_1(X) = X$ here.*

Tangent Variety

Definition ((Relative) Tangent Star)

Let $Y \subseteq X \subseteq \mathbb{P}^n$ *be projective varieties, and let* $p \in Y$*. Let* $\{x_i\}_{i=1}^{\infty}$ *and* $\{y_i\}_{i=1}^{\infty}$ *be sequences having the limit:*

$$\lim_{i\to\infty} x_i = \lim_{i\to\infty} y_i = p \ . \tag{13}$$

The **relative tangent star** *at the point* p *is the union of all limit lines of sequences* x_i *and* y_i *and denoted by* $T_p^*(X, Y)$*. That is to say:*

$$T_p^*(X, Y) = \lim_{i\to\infty} \overline{x_i y_i} \subseteq \mathbb{P}^n \ , \tag{14}$$

the lim *here represents the union of all possible limits and differs from the definition of limits of functions. When* $Y = X$*, then* $T_p^*(X, X)$ *is called* **tangent star** *of* X *at* p *and denoted by* $T_p^*(X)$*.*

Tangent Variety

Definition ((Relative) tangent star variety)

Let $Y \subseteq X \subseteq \mathbb{P}^n$ be projective varieties. The **relative tangent star variety** *of X and Y, usually denoted by $\mathbb{T}(X, Y)$ is defined as:*

$$\mathbb{T}(X, Y) = \bigcup_{p \in Y} T_p^*(X, Y) \ . \tag{15}$$

When $X = Y$, then $\mathbb{T}(X, X)$ is called **tangent variety** *of X and denoted by $\tau(X)$.*

Definition

*The $k-$***tangent variety** *is defined as the relative tangent star variety of $\tau_{k-1}(X)$ and X and is denoted by $\tau_k(X)$. That is to say:*

$$\tau_k(X) = \mathbb{T}(\tau_{k-1}(X), X) \ . \tag{16}$$

Quantum States Postulates

Postulate (Quantum state postulate)

The quantum states of an isolated physical system are represented, at a fixed time by a vector $|\psi\rangle$ *in a topologically separable complex Hilbert space, usually denoted by* $\mathbb{H}$ *and called* **(quantum) Hilbert space**. *The inner product of two vectors* $|\psi\rangle, |\varphi\rangle$ *in the Hilbert space is denoted by* $\langle\psi|\varphi\rangle$.

Postulate (Composite system postulate)

The Hilbert space of a composite system is the tensor product of the Hilbert spaces associated with each subsystem.

Quantum States Postulates

Example (Two-energy-level atom and a photon)

Subsystem 1: *The two-energy-level atom can be described by a Hilbert space $\mathbb{H}_1$ corresponding to its two possible energy states:*

$$\mathbb{H}_1 = \text{Span}\{|0\rangle_{atom}, |1\rangle_{atom}\}\ . \tag{17}$$

$|0\rangle_{atom}$ *represents the ground state (no photon).*
$|1\rangle_{atom}$ *represents the excited state.*

Subsystem 2: *The photon in a cavity can be described by a Hilbert space $\mathbb{H}_2$ corresponding to its two possible energy states:*

$$\mathbb{H}_2 = \text{Span}\{|0\rangle_{photon}, |1\rangle_{photon}\}\ . \tag{18}$$

$|0\rangle_{photon}$ *represents the vacuum state.*
$|1\rangle_{photon}$ *represents the single-photon state.*

Quantum States Postulates

Example (Two-energy-level atom and a photon)

The total system consists of the two subsystems (the atom and the photon), so the Hilbert space of the composite system is:

$$\mathbb{H}_{total} = \mathbb{H}_1 \otimes \mathbb{H}_2 \ . \tag{19}$$

The total Hilbert space is spanned by the following basis states:

- $|0\rangle_{atom} \otimes |0\rangle_{photon}$: *The atom is in the ground state, and there is no photon in the cavity (vacuum state).*
- $|0\rangle_{atom} \otimes |1\rangle_{photon}$: *The atom is in the ground state, and there is one photon in the cavity.*
- $|1\rangle_{atom} \otimes |0\rangle_{photon}$: *The atom is in the excited state, and there is no photon in the cavity.*
- $|1\rangle_{atom} \otimes |1\rangle_{photon}$: *The atom is in the excited state, and there is one photon in the cavity.*

Operator and Measurement Postulates

Postulate (Operator postulate)

A measurable physical quantity is described by a Hermitian operator $\mathcal{Q} : \mathbb{H} \to \mathbb{H}$. The eigenvectors for $\mathcal{Q}$ form an orthogonal basis for $\mathbb{H}$. The result of measuring a physical quantity $\mathcal{Q}$ must be one of the eigenvalues of the corresponding observable $\mathcal{Q}$.

Postulate (Measurement postulate)

The result of measuring a physical quantity $\mathcal{Q}$ must be one of the eigenvalues of the corresponding observable $\mathcal{Q}$. Let λ_i be the eigenvalue of the unit eigenvector $|\psi_i\rangle$, where $i = 1, \cdots, d$. The probability of obtaining λ_n by measuring the quantum state $|\psi_n\rangle$ is given by:

$$\frac{|\langle\psi_n|\psi\rangle|^2}{|\langle\psi_1|\psi\rangle|^2 + |\langle\psi_2|\psi\rangle|^2 + \cdots + |\langle\psi_d|\psi\rangle|^2}. \tag{20}$$

Operator and Measurement Postulates

Example (Cold hydrogen atom)

Consider a cold single hydrogen atom. The electron in the atom can occupy different energy levels. Assume the hydrogen atom is cold enough so that we can only focus on the two lowest energy levels:

$$E_0 = 0 \quad \textit{(ground state)} \quad \textit{and} \quad E_1 = 10.2\, eV \quad \textit{(first excited state)}\,,$$

these energies are obtained from solving the **Schrodinger equation**. *Eigenvectors for the Hamiltonian operator* $\mathcal{H}$ *for this system:*

$$\mathcal{H}\,|\psi_0\rangle = E_0\,|\psi_0\rangle \ , \quad \mathcal{H}\,|\psi_1\rangle = E_1\,|\psi_1\rangle \ . \tag{21}$$

The Hilbert space $\mathbb{H}$ *is spanned by these two states:*

$$\mathbb{H} = \mathrm{Span}\{|\psi_0\rangle\,, |\psi_1\rangle\}\ . \tag{22}$$

Operator and Measurement Postulates

Example (Cold hydrogen atom)

Now, consider the atom in a superposition state:

$$|\psi\rangle = a_0 |\psi_0\rangle + a_1 |\psi_1\rangle \ , \tag{23}$$

where a_0 and a_1 are complex coefficients.
When we measure the energy of the system, the possible outcomes and their probabilities are:

$$E_0 = 0\, eV \, , \quad \textit{with probability } |a_0|^2 \, , \tag{24}$$

$$E_1 = 10.2\, eV \, , \quad \textit{with probability } |a_1|^2 \, . \tag{25}$$

Operator and Measurement Postulates

Example (Cold hydrogen atom in the cavity)

System 1: *A single electron in a cold hydrogen atom with only two energy levels:*

$$E_0 = 0 \quad \textit{(ground state)} \quad \textit{and} \quad E = 10.2\,eV \quad \textit{(first excited state).}$$

System 2: *A photon in a cavity with two possible energy states:*

$$E'_0 = 0 \quad \textit{(no photon)} \quad \textit{and} \quad E' = 2.5\,eV \quad \textit{(single photon).}$$

The Hilbert space of the composite system is the tensor product of the two subsystems, written as:

$$\mathbb{H}_{total} = \mathbb{H}_1 \otimes \mathbb{H}_2 \ . \tag{26}$$

Operator and Measurement Postulates

Example (Cold hydrogen atom in the cavity)

The total energy operator for this composite system, $\mathcal{H}_{total}$, combines the energies of the two systems:

$$\mathcal{H}_{total} = \mathcal{H}_1 \otimes \mathcal{I}_2 + \mathcal{I}_1 \otimes \mathcal{H}_2 \ , \tag{27}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are identity operators for the respective subsystems. The composite Hilbert space is spanned by the total energy eigenstates, and the eigenvalues are:

$$|\psi_{00}\rangle = |\psi_0\rangle_1 \otimes |\psi_0\rangle_2 \, , \quad \textit{with energy } 0\,eV \, , \tag{28}$$

$$|\psi_{01}\rangle = |\psi_0\rangle_1 \otimes |\psi_1\rangle_2 \, , \quad \textit{with energy } 2.5\,eV \, , \tag{29}$$

$$|\psi_{10}\rangle = |\psi_1\rangle_1 \otimes |\psi_0\rangle_2 \, , \quad \textit{with energy } 10.2\,eV \, , \tag{30}$$

$$|\psi_{11}\rangle = |\psi_1\rangle_1 \otimes |\psi_1\rangle_2 \, , \quad \textit{with energy } 12.7\,eV \, . \tag{31}$$

Operator and Measurement Postulates

Example (Cold hydrogen atom in the cavity)

Consider the total system in a superposition state:

$$|\Psi\rangle = a_{00}\,|\psi_{00}\rangle + a_{01}\,|\psi_{01}\rangle + a_{10}\,|\psi_{10}\rangle + a_{11}\,|\psi_{11}\rangle \; . \tag{32}$$

When measuring the total energy, the possible outcomes and their probabilities are:

$$0\,eV\,, \quad \text{with probability } |a_{00}|^2\,, \tag{33}$$
$$2.5\,eV\,, \quad \text{with probability } |a_{01}|^2\,, \tag{34}$$
$$10.2\,eV\,, \quad \text{with probability } |a_{10}|^2\,, \tag{35}$$
$$12.7\,eV\,, \quad \text{with probability } |a_{11}|^2\,, \tag{36}$$

where a_{00}, a_{01}, a_{10}, and a_{11} are dependent on the surrounding and the probabilities are normalized such that $|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$.

Entangled States

- Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be the Hilbert spaces corresponding to two quantum systems, with bases $\{|0\rangle_1, |1\rangle_1\}$ and $\{|0\rangle_2, |1\rangle_2\}$, respectively.
- Consider a quantum state in the composite system $\mathbb{H}_1 \otimes \mathbb{H}_2$ given by:

$$|\mathrm{EPR}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_1 \otimes |0\rangle_2 + |1\rangle_1 \otimes |1\rangle_2 \right). \tag{37}$$

- If a measurement on the first subsystem yields $|0\rangle_1$ (resp. $|1\rangle_1$), then the corresponding measurement on the second subsystem will yield $|0\rangle_2$ (resp. $|1\rangle_2$) with certainty.

Entangled States

- Not all quantum states are entangled states.
- We have a criterion for the entangled states.

Theorem

Let $|\psi\rangle \in \bigotimes_{i=1}^{n} \mathbb{H}_i$. The results of measurement for each subsystem are mutually independent if and only if

$$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle \tag{38}$$

for some $|\phi_i\rangle \in \mathbb{H}_i$.

- We have a mathematical definition of entangled states.

Definition (Entangled and Separable States)

The quantum states having above form are said to be separable. If such a decomposition is not possible, the state $|\psi\rangle$ is said to be entangled.

Projective Geometry and $U(1)$ Gauge Theory

- The quantum states are equivalent in quantum mechanics if $|\varphi\rangle = e^{i\theta} |\psi\rangle$.
- Physically, the stationary quantum mechanics is a $U(1)$ gauge theory.
- Recall that the underlying set of $\mathbb{P}^n_k$ is:

$$\mathbb{P}^n_k = \{[a_1 : \cdots : a_{n+1}] \mid [a_1, \cdots, a_{n+1}] \in k^{n+1} \backslash \{0\}\}, \tag{39}$$

where

$$[a_1 : \cdots : a_{n+1}] = [b_1 : \cdots : b_{n+1}] \iff c[a_1, \cdots, a_{n+1}] = [b_1, \cdots, b_{n+1}] \text{ for some } c \in k^* . \tag{40}$$

Separable States for Distinguishable Particles

- Separable quantum states are simple tensors in the Hilbert space:

$$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle \,. \tag{41}$$

- We can use the Segre variety to describe the separable states.

Theorem

Let an n-distinguishable-particle system have the Hilbert space $\mathbb{H} = \bigotimes_{i=1}^{n} \mathbb{H}_i$. Then the set of all separable states forms a Segre variety:

$$S \subseteq \mathbb{P}(\mathbb{H}). \tag{42}$$

Separable States for Bosons: Veronese Varieties

- Bosons obey Bose-Einstein statistics; their states are symmetric.
- Hilbert space for bosonic systems: $\mathrm{Sym}^n\mathbb{H}$.
- Coherent states:

Definition

A quantum state $|\varphi\rangle \in \mathrm{Sym}^n\mathbb{H}$ is said to be *coherent* if there exists $|\psi\rangle \in \mathbb{H}$ such that:

$$|\varphi\rangle = |\psi\rangle^{\odot n} \ . \tag{43}$$

- We can use the Veronese variety to describe the coherent states for bosons.

Theorem

The set of all separable states forms a Veronese variety:

$$V \subseteq \mathbb{P}(Sym^n\mathbb{H}). \tag{44}$$

Separable States for Fermions: Grassmann Varieties

- Fermions obey Fermi-Dirac statistics; their states are antisymmetric.
- Hilbert space for fermionic systems: $\bigwedge^n \mathbb{H}$.
- We can use the Grassmann variety to describe the coherent states for fermions.

Theorem

The set of all separable states forms a Grassmann variety:

$$G \subseteq \mathbb{P}(\bigwedge^n \mathbb{H}). \tag{45}$$

Entangled States, Secant, and Tangent Varieties

- Secant varieties generalize separability and describe entanglement.
- Inclusion hierarchy:

$$X = \sigma_1(X) \subseteq \sigma_2(X) \subseteq \sigma_3(X) \subseteq \cdots \subseteq \mathbb{P}(\mathbb{H}). \quad (46)$$

- Larger k corresponds to higher levels of entanglement.
- Tangent varieties are subsets of secant varieties:

$$\tau_k(X) \subseteq \sigma_k(X). \quad (47)$$

- Tangent and secant varieties provide geometric measures of entanglement.

Summary

Quantum mechanics	Algebraic geometry
$U(1)$ gauge theory	Projective space
Entanglement classes	Projective varieties
Separable states	Segre varieties
Coherent states for bosons	Veronese varieties
Coherent states for fermions	Grassmann varieties
Entangled states	Secant and tangent varieties
Degree of entanglement	Indices of proper secant varieties

Table: Relations between quantum mechanics and algebraic geometry.

Entangled States Classification Theorem

Theorem (Entangled States Classification)

Quantum states in a Hilbert space $\mathbb{H}$ can be classified as follows:

- *Identify the Segre, Veronese, or Grassmann variety $X \subseteq \mathbb{P}(\mathbb{H})$.*
- *Classify entangled states by the sequence of k-secant varieties:*

$$X = \sigma_1(X) \subseteq \sigma_2(X) \subseteq \cdots \subseteq \sigma_n(X) \subseteq \mathbb{P}(X). \quad (48)$$

  Each $\varsigma_k(X) = \sigma_k(X) \setminus \sigma_{k-1}(X)$ represents an entangled state family.
- *Use k-tangent varieties $\tau_k(X) \subseteq \sigma_k(X)$ to refine classification:*

  If $\tau_k(X) \subsetneq \sigma_k(X)$, then $\tau_k(X)$ and $\varsigma_k(X) \setminus \tau_k(X)$ form distinct families.

Entangled States Classification Theorem

Theorem (Entangled States Classification)

- *Summarize entangled state families in a table:*

Entangled State Families	*Varieties*
Class N	$\tau_k(X)$
Class $N-1$	$\varsigma_k(X) \setminus \tau_k(X)$
$\vdots$	$\vdots$
Class 1	X

Distinguishable two-qubit system

Example (Distinguishable two-qubit system)

Consider a two-qubit quantum system with Hilbert space with Hilbert space $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_1 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$*, where* $\mathbb{H}_1$ *having orthonormal basis* $\{|0\rangle, |1\rangle\}$*. The projective Hilbert space is* $\mathbb{P}(\mathbb{H}) \cong \mathbb{P}^3_{\mathbb{C}}$*. The Segre variety for* $\mathbb{P}^3_{\mathbb{C}}$ *is given by:*

$$X = \left\{ \left[1 : a : b : ab \right] \in \mathbb{P}(\mathbb{H}) \,\middle|\, a, b \in \mathbb{C} \right\}. \tag{49}$$

We apply the proper secant variety first. The $2-$*secant variety of* X *is given by:*

$$\sigma_2(X) = \left\{ \left[\lambda_1 + \lambda_2 : \lambda_1 a_1 + \lambda_2 a_2 : \lambda_1 b_1 + \lambda_2 b_2 : \lambda_1 a_1 b_1 + \lambda_2 a_2 b_2 \right] \in \mathbb{P}(\mathbb{H}) \,\middle|\, \lambda_1, \lambda_2, a, b \in \mathbb{C} \right\}. \tag{50}$$

Distinguishable two-qubit system

Example (Distinguishable two-qubit system)

The 2−secant variety fills all $\mathbb{P}(\mathbb{H}) \cong \mathbb{P}^3_{\mathbb{C}}$*. We can conclude the sequence:*

$$X = \sigma_1(X) \subseteq \sigma_2(X) = \mathbb{P}^3_{\mathbb{C}} \cong \mathbb{P}(\mathbb{H}) \,. \quad (51)$$

The 2−tangent star variety of X *is given by:*

$$\tau_2(X) = \bigcup_{p \in X} T^*_p(X) \,, \quad (52)$$

where the tangent star at $p = [1 : a : b : ab] \in X$ *can be calculated as follows:*

$$T^*_p(X) = \left\{ \lim_{\epsilon \to 0} \frac{\lambda}{\epsilon}([1 : a + \epsilon : b + \epsilon : (a + \epsilon)(b + \epsilon)] - [1 : a : b : ab]) + [1 : a \right.$$
$$= \{\lambda[0 : 1 : 1 : a + b] + [1 : a : b : ab] \mid \lambda \in \mathbb{C}\} \,. \quad (53)$$

Distinguishable two-qubit system

Example (Distinguishable two-qubit system)

We can conclude that:

$$\tau_2(X) = \bigcup_{p \in X} T_p^*(X) = \mathbb{P}^3_{\mathbb{C}} = \sigma_2(X) \cong \mathbb{P}(\mathbb{H}) \,. \tag{54}$$

So we can make the following table:

Names	*Varieties*
Bell states	$\varsigma_2(X)$
Separable states	X

Table: Table for the entanglement families, their corresponding varieties.

Bosonic three-qubit system

Example (Bosonic three-qubit system)

Consider a two-qubit quantum system with Hilbert space with Hilbert space $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_1 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$, where $\mathbb{H}_1$ having orthonormal basis $\{|0\rangle, |1\rangle\}$. The projective Hilbert space is $\mathbb{P}(\mathbb{H}) \cong \mathbb{P}^3_{\mathbb{C}}$. The Segre variety for $\mathbb{P}^3_{\mathbb{C}}$ is given by:

$$X = \left\{ \left[1 : a : b : ab \right] \in \mathbb{P}(\mathbb{H}) \;\middle|\; a, b \in \mathbb{C} \right\}. \tag{55}$$

We apply the proper secant variety first. The $2-$secant variety of X is given by:

$$\sigma_2(X) = \left\{ \left[\lambda_1 + \lambda_2 : \lambda_1 a_1 + \lambda_2 a_2 : \lambda_1 b_1 + \lambda_2 b_2 : \lambda_1 a_1 b_1 + \lambda_2 a_2 b_2 \right] \in \mathbb{P}(\mathbb{H}) \;\middle|\; \lambda_1, \lambda_2, a, b \in \mathbb{C} \right\}. \tag{56}$$

Bosonic three-qubit system

Example (Bosonic three-qubit system)

We can see that $\sigma_2(X) = \mathbb{P}(\mathrm{Sym}^3 \mathbb{H}) \cong \mathbb{P}^3_{\mathbb{C}}$*. Conclude the sequence:*

$$V = \sigma_1(V) \subseteq \sigma_2(V) = \mathbb{P}(\mathrm{Sym}^3 \mathbb{H}) \, . \tag{57}$$

Hence, we can conclude The $2-$*tangent variety can be calculated by considering a point* $p = [1 : x : x^2 : x^3] \in V$*. The tangent for* p *is:*

$$\begin{aligned} T_p^*(V) &= \left\{ \lim_{\epsilon \to 0} \frac{\lambda}{\epsilon}([1 : (x+\epsilon) : (x+\epsilon)^2 : (x+\epsilon)^3] \right. \\ &\qquad \left. - [1 : x : x^2 : x^3]) + p \,\middle|\, \lambda \in \mathbb{C} \right\} \\ &= \{[2 : \lambda + x : 2\lambda x + x^2 : 3\lambda x^2 + x^3] \mid \lambda \in \mathbb{C}\} \, . \end{aligned} \tag{58}$$

Bosonic three-qubit system

Example (Bosonic three-qubit system)

Hence, the tangent star variety is:

$$\tau_2(V) = \bigcup_{p \in V} T_p^*(V) = \left\{ [2 : \lambda + x : 2\lambda x + x^2 : 3\lambda x^2 + x^3] \mid x, \lambda \in \mathbb{C} \right\} . \tag{59}$$

It is easy to see that $\tau_2(V) \subsetneq \sigma_2(V)$*, so we can make the following table:*

Names	*Varieties*
W states	$\tau_2(V)$
GHZ states	$\varsigma_2(V) \backslash \tau_2(V)$
Separable states	V

Table: Table for the entanglement families, their corresponding varieties.