Projective Geometry Description and Applications for Quantum Entanglement

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Abstract

Quantum entanglement, which is a pure-quantum phenomenon that describes the distance-free interaction between quantum systems and does not have a corresponding classical theory, is an era-breaking phenomenon that even won the Nobel Physics Prize. Even if quantum entanglement has been observed and verified experimentally, the mathematical foundation for the description of various kinds of multipartite entanglement is still incredibly difficult. Algebraic geometry tools, such as projective geometry and varieties, have been applied to study this mysterious phenomenon and found that there is a useful mathematical description. We discuss some useful algebraic geometry tools and theorems that can describe quantum entanglement and how they be applied to some quantum entanglement systems.

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1 Variety

1.1 Segre Variety

The Segre embedding is a significant concept in projective geometry that allows us to embed the product of two projective spaces into a higher-dimensional projective space. It is an example of a geometric construction that reflects the combinatorial nature of tensor products in algebraic geometry.

Definition 1 (Segre embedding)**.** *The Segre embedding is an injective morphism defined as follows:*

$$\Sigma : \mathbb{P}_k^{d_1-1} \times \mathbb{P}_k^{d_2-1} \hookrightarrow \mathbb{P}_k^{d_1 d_2-1} \ , \tag{1.1}$$

which takes a pair of points $([a],[b]) \in \mathbb{P}_k^{d_1-1} \times \mathbb{P}_k^{d_2-1}$ *to their products:*

$$([a_0 : \cdots : a_{d_1-1}],[b_0 : \cdots : b_{d_2-1}]) \mapsto [a_0 b_0 : a_0 b_1 : \cdots : a_{d_1-1} b_{d_2-1}] \ , \tag{1.2}$$

where the notation refers to homogeneous coordinates, and the $a_i b_j$ *are ordered in lexicographical order. The map is well-defined in terms of the projective coordinates, and the resulting set of points forms a variety within the higher-dimensional projective space* $\mathbb{P}_k^{d_1 d_2-1}$*.*

Definition 2 (Segre variety)**.** *The image of the Segre embedding is a variety, called the* ***Segre variety****.*

Remark 1. *In linear algebra, for given vector spaces* U *and* V *over the same field* K*, there is a natural bilinear map:*

$$\varphi : U \times V \to U \otimes V, \quad (u,v) \mapsto u \otimes v \ . \tag{1.3}$$

It is easy to see that for $u \in U$*,* $v \in V$*, and any nonzero* $c \in K$*, we have:*

$$\varphi(u,v) = u \otimes v = cu \otimes c^{-1} v = \varphi(cu, c^{-1} v) \ . \tag{1.4}$$

since $(u,v) \neq (cu, c^{-1}v)$ *in general,* φ *is not injective in general.*

However, considering the underlying projective spaces $\mathbb{P}(U)$ *and* $\mathbb{P}(V)$*, this map becomes an injective morphism of varieties:*

$$\mathbb{P}(U) \times \mathbb{P}(V) \to \mathbb{P}(U \otimes V), \quad ([u],[v]) \mapsto [u \otimes v] \ . \tag{1.5}$$

This can be easily extended to the tensor product of n *spaces:*

$$\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n) \to \mathbb{P}\left(\bigotimes_{i=1}^{n} V_i\right), \quad ([v_1], \cdots, [v_n]) \mapsto [v_1 \otimes \cdots \otimes v_n] \ . \tag{1.6}$$

Remark 1 shows that the projective map induced by the tensor product is injective, and the image of this map is the Segre variety, a well-defined variety in the projective space $\mathbb{P}(U \otimes V)$. Hence, $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n)$ can be considered as a subvariety in $\mathbb{P}(U \otimes V)$.

1.2 Veronese Variety

Since we have discussed that the Segre varieties can be considered as the tensor product of two projective spaces. We can consider the symmetric and antisymmetric tensor products as well. Hence, there are still two kinds of varieties that can be considered as the extension of the Segre variety. The first one is called *Veronese variety.*

Definition 3 (Veronese embedding)**.** *The degree n Veronese embedding*

$$\Omega : \mathbb{P}^d_k \hookrightarrow \mathbb{P}^m_k \ , \tag{1.7}$$

is an injective morphism defined by:

$$[a_0 : \cdots : a_d] \mapsto [f_0(a_0, \cdots, a_n) : \cdots : f_m(a_0, \cdots, a_n)] \ , \tag{1.8}$$

where f_i, $i = 0 \cdots, m$ are all of the d-variate degree-n monomials.

Definition 4 (Veronese variety)**.** *The image of the Veronese embedding is a variety, called the* ***Veronese variety****.*

Remark 2. *As the discussion in remark 1. By considering the underlying projective space $\mathbb{P}(V)$, this map becomes an injective morphism of varieties:*

$$\mathbb{P}(V) \to \mathbb{P}(V \odot V), \quad [v] \mapsto [v \odot v] \ . \tag{1.9}$$

Definitely, as the tensor product case, we can consider the symmetric tensor product for more spaces as well:

$$\mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^m V), \quad [v] \mapsto [v^{\odot m}] \ . \tag{1.10}$$

Similar to the tensor product case discussed in remark 1. Since the degree n Veronese embedding is injective, we may consider $\mathbb{P}(V)$ as a subvariety of $\mathbb{P}(\mathrm{Sym}^m V)$.

1.3 Grassmann Variety

Now we can consider the antisymmetric tensor products. This is called *Grassmann variety.* Similar to previous ones, we define an embedding first.

Definition 5 (Plucker embedding)**.** *Let V be a finite-dimensional vector space over k and denote the Grassmannian by $\mathbb{G}(d, V)$. The degree d Plucker embedding*

$$\mathbb{G}(d, V) \hookrightarrow \mathbb{P}_k^{\binom{\dim V}{d} - 1} \ , \tag{1.11}$$

is an injective morphism defined by:

$$W \mapsto [w_1 \wedge \cdots \wedge w_d], \quad \textit{where } \{w_1, \cdots, w_n\} \textit{ is a basis for } W \subseteq V \ . \tag{1.12}$$

Definition 6 (Grassmann variety)**.** *The image of the Plucker embedding is a variety, called the* ***Grassmann variety****.*

Remark 3. *As the discussion in remark 1. By considering the underlying Grassmannian $\mathbb{G}(d, V)$, this map becomes an injective morphism of varieties:*

$$\begin{aligned} &\mathbb{G}(d, V) \to \mathbb{P}\left(\bigwedge^d V\right) , \\ &W \mapsto [w_1 \wedge \cdots \wedge w_d], \quad \textit{where } \{w_1, \cdots, w_d\} \textit{ is a basis for } W \subseteq V . \end{aligned} \tag{1.13}$$

Similarly, since the Plucker embedding is injective, the abstract Grassmannian can be considered as a subvariety of $\mathbb{P}\left(\bigwedge^d V\right)$ and gives another way for constructing the Grassmannian.

1.4 Secant and Tangent Varieties

Just like we have secants and tangents in differential geometry, we also want to extend the notions of secants and tangents to algebraic geometry.

1.4.1 Secant Variety

In algebraic geometry, a secant variety is a construction that extends the notion of "linear span" to a collection of points on a projective variety. It is particularly important in the classification of quantum entanglement because it describes states that can be expressed as superpositions of separable states. We start our discussion from the projective lines in projective geometry.

Definition 7 (Projective line)**.** *Let $A = [a_0 : \cdots : a_n]$ and $B = [b_0 : \cdots : b_n] \in \mathbb{P}^n$ be two distinct points in $\mathbb{P}^n$, they correspond to two vectors $\mathbf{a} = (a_0, \ldots, a_n)$ and $\mathbf{b} = (b_0, \ldots, b_n) \in k^{n+1}$, which span a plane $\Lambda \subset k^{n+1}$, whose vectors are of the form $u\mathbf{a} + v\mathbf{b}$ with $u, v \in k$.*

The corresponding line $\overline{AB} = \mathbb{P}(\Lambda) \subset \mathbb{P}^n$ is:

$$\mathbb{P}(\Lambda) = \{[ua_0 + vb_0 : \cdots : ua_n + vb_n] \mid u, v \in k, \ \textit{not both zero}\} . \tag{1.14}$$

Example 1.1. *The projective line $\overline{pq}$ joining $p = [1 : 0 : 0 : 0]$ and $q = [a : b : c : d]$ has points with coordinates:*

$$[u + va : vb : vc : vd] . \tag{1.15}$$

Definition 8. *(Secant variety)*
Let projective varieties X and Y be subvarieties of a projective variety. The joining of X and Y is given by the Zariski closure, of the lines from one to the other,

$$\mathbb{J}(X, Y) = \overline{\bigcup_{x \in X, y \in Y} \overline{xy}} \tag{1.16}$$

where $\overline{xy}$ is the projective line that includes both x and y called a ***chord*** *or a* ***secant line****. If $X = Y$, then the variety is called a* ***secant variety of*** *X and denoted by $\sigma(X)$, that is,*

$$\sigma(X) = \mathbb{J}(X, X) = \overline{\bigcup_{x, y \in X} \overline{xy}} . \tag{1.17}$$

Now we want to extend the notion to multiple-point cases that pass through k points of X.

Definition 9. *Let $X \subset \mathbb{P}^N$ be a projective variety. The $k-$secant variety of X, denoted $\sigma_k(X)$, is the Zariski closure of the union of all projective linear spaces spanned by $k+1$ points on X. In other words, $\sigma_k(X)$ consists of all points in $\mathbb{P}^N$ that lie on secant k-planes of X:*

$$\sigma_k(X) = \overline{\bigcup_{p_0,\ldots,p_k \in X} \langle p_0, \ldots, p_k \rangle}\ , \tag{1.18}$$

where $\langle p_0, \ldots, p_k \rangle$ denotes the projective span of the points $p_0, \ldots, p_k$.

Theorem 1.1. *If the projective variety $X \subseteq \mathbb{P}(k^d)$ is non-degenerate, i.e., it is not contained in a linear subspace of $\mathbb{P}(k^d)$, there is a sequence of inclusions given by:*

$$X \subseteq \sigma(X) \subseteq \sigma_3(X) \subseteq \cdots \subseteq \sigma_N(X) = \mathbb{P}(k^d)\ , \tag{1.19}$$

where N is the smallest integer such that the N^{th} secant variety fills the ambient space.

Remark 4. *Due to the sequence given in the theorem 1.1, the variety $\sigma(X)$ is indexed as $\sigma_2(X)$, and $\sigma_1(X)$ is defined as X. So the sequence in the theorem 1.1 becomes:*

$$X = \sigma_1(X) \subseteq \sigma_2(X) \subseteq \cdots \subseteq \sigma_N(X) = \mathbb{P}(k^d)\ , \tag{1.20}$$

With theorem 1.1, we can naturally define another important variety.

Definition 10. *Let $\sigma_k(X)$ and $\sigma_{k-1}(X)$ be varieties in the inclusion sequence:*

$$X = \sigma_1(X) \subseteq \sigma_2(X) \subseteq \cdots \subseteq \sigma_N(X) = \mathbb{P}(k^d)\ , \tag{1.21}$$

then the ***proper $k-$secant variety of*** *X, denoted by $\varsigma_k(X)$ is:*

$$\varsigma_k = \sigma_k(X) \backslash \sigma_{k-1}(X)\ . \tag{1.22}$$

In the context of quantum entanglement, secant varieties of the Segre variety play a central role. The Segre variety X represents separable states in multipartite quantum systems. The k-th secant variety $\sigma_k(X)$ corresponds to states that can be expressed as superpositions of $k+1$ separable states. For example, in a bipartite system, $\sigma_1(X)$ includes states of Schmidt rank at most 2, while higher k-secant varieties describe states with increasing levels of entanglement complexity. Thus, secant varieties provide a natural geometric framework for classifying entanglement in quantum systems. We will discuss more details about it in the following sections.

1.4.2 Tangent Star Variety

At a smooth point, it is natural to define the tangent space. However, in algebraic geometry, when we include the singular point, there are several different definitions of tangent space. We discuss the most useful one in quantum mechanics-the tangent star.

Definition 11 ((Relative) Tangent Star)**.** *Let $Y \subseteq X \subseteq \mathbb{P}^n$ be projective varieties, and let $p \in Y$. Let $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ be sequences having the limit:*

$$\lim_{i\to\infty} x_i = \lim_{i\to\infty} y_i = p\ . \tag{1.23}$$

*The **relative tangent star** at the point p is the union of all limit lines of sequences x_i and y_i and denoted by $T_p^*(X,Y)$. That is to say:*

$$T_p^*(X,Y) = \lim_{i\to\infty} \overline{x_i y_i} \subseteq \mathbb{P}^n \ , \tag{1.24}$$

the lim *here represents the union of all possible limits and differs from the definition of limits of functions. When $Y = X$, then $T_p^*(X,X)$ is called **tangent star** of X at p and denoted by $T_p^*(X)$.*

The tangent star is a geometric concept closely related to the study of tangent spaces to a variety but involves the collection of all tangent lines to the variety through a given point. It is particularly useful when analyzing the local geometric structure of a variety near a point.

Example 1.2. *Let L_1 and L_2 be two lines in $\mathbb{P}^2_{\mathbb{R}}$ intersecting at a point p that are not contained in each other. Let $X = L_1 \bigcup L_2 \subseteq \mathbb{P}^2_{\mathbb{R}}$. Then the tangent star of X at point p is:*

$$T_p^*(X) = \mathrm{Span}\{L_i, L_j\} \subseteq \mathbb{P}^n_{\mathbb{R}} \ . \tag{1.25}$$

The tangent variety of a given algebraic variety is a construction that captures information about the tangents of the variety. It is defined as the union of all tangent spaces to the variety at its smooth points. Tangent varieties are particularly useful in studying the geometry of a variety and its embedding in a projective or affine space.

Definition 12 ((Relative) tangent star variety)**.** *Let $Y \subseteq X \subseteq \mathbb{P}^n$ be projective varieties. The **relative tangent star variety** of X and Y, usually denoted by $\mathbb{T}(X,Y)$ is defined as:*

$$\mathbb{T}(X,Y) = \bigcup_{p\in Y} T_p^*(X,Y) \ . \tag{1.26}$$

*When $X = Y$, then $\mathbb{T}(X,X)$ is called **tangent variety** of X and denoted by $\tau(X)$.*

As secant varieties, we also want to extend tangent variety to multiple-point cases that pass through k points of X.

Definition 13. *The $k-$**tangent variety** is defined as the relative tangent star variety of $\tau_{k-1}(X)$ and X and is denoted by $\tau_k(X)$. That is to say:*

$$\tau_k(X) = \mathbb{T}(\tau_{k-1}(X), X) \ . \tag{1.27}$$

Remark 5. *Due to the sequence given in remark 4, we define $\tau(X) = \tau_2(X)$ and $\tau_1(X) = X$.*

Remark 6. *From now on, we call the tangent star variety **tangent variety** for simplicity.*

2 Quantum Mechanics

2.1 Postulates in Quantum Mechanics

Quantum mechanics is a theory that differs from classical mechanics in that Newton's laws break the framework. To reformulate the physics in quantum mechanics, physicists proposed several quantum mechanics postulates, which are experimentally verified but still not strictly "proved" so far, to explain the phenomena that classical theory cannot explain.

2.1.1 Quantum States

From basic mechanics knowledge, we know that we need two quantities, the position x and momentum p, and the pair (x, p) which is called the *state*, to describe a mechanical system. From these two quantities and three Newton's laws, we can describe the evolution of states for a so-called mechanical system.

This description needed to be changed to accommodate experimental results. For example, in Einstein's space-time deformation and geodesic motion, we describe the state by a so-called Lorentz tensor $g_{\mu\nu}$. In classical field theory, we use electric field, magnetic field, etc., to describe the states.

Since the most fundamental notion in a physical system is the state of the system, the first postulate in quantum mechanics, the most fundamental postulate in quantum mechanics, provides us with a description of quantum states.

Postulate 1 (Quantum state postulate)**.** *The quantum states of an isolated physical system are represented, at a fixed time by a vector $|\psi\rangle$ in a topologically separable complex Hilbert space, usually denoted by $\mathbb{H}$ and called* ***(quantum) Hilbert space****. The inner product of two vectors $|\psi\rangle, |\varphi\rangle$ in the Hilbert space is denoted by $\langle\psi|\varphi\rangle$.*

Remark 7. *Notice that $C|\psi\rangle$, $C \neq 0$, and $|\psi\rangle$ represent the same physical state. However, $C = |C|e^{i\theta}$ does not imply the phase has physical meaning; see more details on quantum dynamics. However since we only discuss the stationary quantum mechanics in this note, we will not discuss this deeper notion here.*

From this postulate, although we can describe a quantum state by a vector. One may still be confused with what the Hilbert space of a quantum system looks like, we will discuss this problem soon.

The postulate only provides us with a way to describe the isolated systems, that is, the systems without any interaction with others. Another postulate regarding the quantum states provides us with a way to describe the quantum states in composite quantum systems.

Postulate 2 (Composite system postulate)**.** *The Hilbert space of a composite system is the tensor product of the Hilbert spaces associated with each subsystem.*

Unlike classical mechanics, the interaction between quantum systems can not be described by force and collision. Fortunately, by this postulate, the description of composite quantum systems provides us with a framework to explain the interaction between two quantum systems from the quantum mechanics point of view. We will see more details later.

Example 2.1 (Two-energy-level atom and a photon (quantum optical system))**.** *Imagine two simple quantum systems:*

- *Subsystem 1: The two-energy-level atom can be described by a Hilbert space $\mathbb{H}_1$ corresponding to its two possible energy states:*

$$\mathbb{H}_1 = \mathrm{Span}\{|0\rangle_{atom}, |1\rangle_{atom}\} . \tag{2.1}$$

  Here, $|0\rangle$ represents the ground state (e.g., the atom is not excited), and $|1\rangle$ represents the excited state (the atom is in an excited energy level).

- *Subsystem 2: The photon in a cavity can be described by a Hilbert space* $\mathbb{H}_2$. *For simplicity, let's assume the photon can either be in the vacuum state* $|0\rangle_{photon}$ *or in a single-photon state* $|1\rangle_{photon}$*, representing the presence or absence of a photon in the cavity:*

$$\mathbb{H}_2 = \text{Span}\{|0\rangle_{photon}, |1\rangle_{photon}\} . \tag{2.2}$$

The total system consists of the two subsystems (the atom and the photon), so the Hilbert space of the composite system is the tensor product of $\mathbb{H}_1$ *and* $\mathbb{H}_2$*:*

$$\mathbb{H}_{total} = \mathbb{H}_1 \otimes \mathbb{H}_2 . \tag{2.3}$$

This space describes all possible states of the combined system, where each component (atom and photon) can be in any of their respective states.

The total Hilbert space will span the following basis states:

$$\mathbb{H}_{total} = \text{Span}\{|0\rangle_{atom} \otimes |0\rangle_{photon}, |0\rangle_{atom} \otimes |1\rangle_{photon}, |1\rangle_{atom} \otimes |0\rangle_{photon}, |1\rangle_{atom} \otimes |1\rangle_{photon}\} . \tag{2.4}$$

These states represent:

- $|0\rangle_{atom} \otimes |0\rangle_{photon}$: *The atom is in the ground state, and there is no photon in the cavity (vacuum state).*
- $|0\rangle_{atom} \otimes |1\rangle_{photon}$: *The atom is in the ground state, and there is one photon in the cavity.*
- $|1\rangle_{atom} \otimes |0\rangle_{photon}$: *The atom is in the excited state, and there is no photon in the cavity.*
- $|1\rangle_{atom} \otimes |1\rangle_{photon}$: *The atom is in the excited state, and there is one photon in the cavity.*

In this example, the atom and the photon are independent subsystems, and the state of the combined system can be described by the tensor product of their individual states. The composite system ($\mathbb{H}_{total}$*) allows us to represent the possible interactions and correlations between the atom and the photon. For instance, when the atom is excited (*$|1\rangle_{atom}$*) and there is a photon in the cavity (*$|1\rangle_{photon}$*), the system could represent an interaction where the atom emits or absorbs a photon.*

The tensor product structure reflects the fact that the total system is described by independent subsystems that can each be in different states. For instance, the atom might be in state $|0\rangle_{atom}$ *(ground state) while the photon is in state* $|1\rangle_{photon}$ *(one-photon state), represented as* $|0\rangle_{atom} \otimes |1\rangle_{photon}$. *These are independent possibilities, so the total system's state is described by the tensor product of the individual states.*

This example illustrates how the Composite System Postulate works in practice. The total Hilbert space of the system (the atom and photon together) is the tensor product of the Hilbert spaces of the individual subsystems (the atom and the photon). The tensor product allows us to describe all the possible combinations of states between the atom and the photon, and it is a key idea when considering composite quantum systems in quantum mechanics.

2.1.2 Physical Quantities and Measurement

Given a quantum system, to describe the quantum system, we need to formulate the associated Hilbert space. In addition, in physics, besides the state of the system, we are also interested in other physical quantities, such as energy and momentum. Hence, we want to calculate the physical quantities, from the constructed Hilbert space in quantum mechanics framework. The following postulates provide us with the answers to these problems.

Postulate 3 (Operator postulate)**.** *A measurable physical quantity is described by a Hermitian operator $\mathcal{Q} : \mathbb{H} \to \mathbb{H}$. The eigenvectors for $\mathcal{Q}$ form an orthogonal basis for $\mathbb{H}$. The result of measuring a physical quantity $\mathcal{Q}$ must be one of the eigenvalues of the corresponding observable $\mathcal{Q}$.*

Postulate 4 (Measurement postulate)**.** *The result of measuring a physical quantity $\mathcal{Q}$ must be one of the eigenvalues of the corresponding observable $\mathcal{Q}$. Let λ_i be the eigenvalue of the unit eigenvector $|\psi_i\rangle$, where $i = 1, \cdots, d$. The probability of obtaining λ_n by measuring the quantum state $|\psi_n\rangle$ is given by:*

$$\frac{|\langle\psi_n|\psi\rangle|^2}{|\langle\psi_1|\psi\rangle|^2 + |\langle\psi_2|\psi\rangle|^2 + \cdots + |\langle\psi_d|\psi\rangle|^2} \; . \tag{2.5}$$

So far, we have the theoretical basis for stationary quantum mechanics. We can now consider some concrete examples.

Example 2.2 (Cold hydrogen atom)**.** *Consider a cold single hydrogen atom. The electron in the atom can occupy different energy levels. For simplicity, assume the hydrogen atom is cold enough so that we can only focus on the two lowest energy levels:*

$$E_0 = 0 \quad \textit{(ground state)} \quad \textit{and} \quad E_1 = 10.2\, eV \quad \textit{(first excited state)}\, ,$$

*these energies can be obtained from solving the well-known **Schrodinger equation**. The Hamiltonian operator $\mathcal{H}$ for this system acts on the quantum states $|\psi_0\rangle$ and $|\psi_1\rangle$, which correspond to the ground state and first excited state, respectively:*

$$\mathcal{H}\,|\psi_0\rangle = E_0\,|\psi_0\rangle \;, \quad \mathcal{H}\,|\psi_1\rangle = E_1\,|\psi_1\rangle \; . \tag{2.6}$$

The Hilbert space $\mathbb{H}$ is spanned by these two states:

$$\mathbb{H} = \mathrm{span}\{|\psi_0\rangle, |\psi_1\rangle\} \; . \tag{2.7}$$

Now, consider the atom in a superposition state:

$$|\psi\rangle = a_0\,|\psi_0\rangle + a_1\,|\psi_1\rangle \;, \tag{2.8}$$

where a_0 and a_1 are complex coefficients.

When we measure the energy of the system, the possible outcomes and their probabilities are:

$$E_0 = 0\, eV\,, \quad \textit{with probability } |a_0|^2 \;, \tag{2.9}$$

$$E_1 = 10.2\, eV\,, \quad \textit{with probability } |a_1|^2 \; . \tag{2.10}$$

This example illustrates a simple quantum system—the hydrogen atom with two energy levels—and connects it to basic quantum measurement principles.

Example 2.3 (Cold hydrogen atom in the cavity)**.** *Imagine two simple quantum systems:*

- *System 1: A single electron in a cold hydrogen atom with only two energy levels:*

$$E_0 = 0 \quad \textit{(ground state)} \quad \textit{and} \quad E = 10.2\, eV \quad \textit{(first excited state)}.$$

- *System 2: A photon in a cavity with two possible energy states:*

$$E'_0 = 0 \quad \textit{(no photon)} \quad \textit{and} \quad E' = 2.5\, eV \quad \textit{(single photon)}.$$

The total quantum system consists of both the electron and the photon. The Hilbert space of the composite system is the tensor product of the two subsystems, written as:

$$\mathbb{H}_{total} = \mathbb{H}_1 \otimes \mathbb{H}_2 \; . \tag{2.11}$$

The total energy operator for this composite system, $\mathcal{H}_{total}$*, combines the energies of the two systems:*

$$\mathcal{H}_{total} = \mathcal{H}_1 \otimes \mathcal{I}_2 + \mathcal{I}_1 \otimes \mathcal{H}_2 \; , \tag{2.12}$$

where $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are identity operators for the respective subsystems.*

The total energy eigenstates and eigenvalues are:

$$|\psi_{00}\rangle = |\psi_0\rangle_1 \otimes |\psi_0\rangle_2 \, , \quad \textit{with energy } 0\, eV \, , \tag{2.13}$$
$$|\psi_{01}\rangle = |\psi_0\rangle_1 \otimes |\psi_1\rangle_2 \, , \quad \textit{with energy } 2.5\, eV \, , \tag{2.14}$$
$$|\psi_{10}\rangle = |\psi_1\rangle_1 \otimes |\psi_0\rangle_2 \, , \quad \textit{with energy } 10.2\, eV \, , \tag{2.15}$$
$$|\psi_{11}\rangle = |\psi_1\rangle_1 \otimes |\psi_1\rangle_2 \, , \quad \textit{with energy } 12.7\, eV \, . \tag{2.16}$$

The composite Hilbert space is spanned by these four states:

$$\mathbb{H}_{total} = \mathrm{span}\{|\psi_{00}\rangle \, , |\psi_{01}\rangle \, , |\psi_{10}\rangle \, , |\psi_{11}\rangle\} \; . \tag{2.17}$$

Now, consider the total system in a superposition state:

$$|\Psi\rangle = a_{00} \, |\psi_{00}\rangle + a_{01} \, |\psi_{01}\rangle + a_{10} \, |\psi_{10}\rangle + a_{11} \, |\psi_{11}\rangle \; . \tag{2.18}$$

When measuring the total energy, the possible outcomes and their probabilities are:

$$0\, eV \, , \quad \textit{with probability } |a_{00}|^2 \; , \tag{2.19}$$
$$2.5\, eV \, , \quad \textit{with probability } |a_{01}|^2 \; , \tag{2.20}$$
$$10.2\, eV \, , \quad \textit{with probability } |a_{10}|^2 \; , \tag{2.21}$$
$$12.7\, eV \, , \quad \textit{with probability } |a_{11}|^2 \; , \tag{2.22}$$

where a_{00}*,* a_{01}*,* a_{10}*, and* a_{11} *are dependent on the surrounding and the probabilities are normalized such that* $|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$*.*

This example illustrates a composite quantum system involving an electron and a photon, linking the abstract mathematical description to a tangible physical scenario.

So far we have discussed the essential formalism for quantum entanglement. For more details about quantum mechanics, see the note here. We discuss the main topic regarding quantum mechanics in this note, that is, the quantum entanglement, directly in the next chapter.

2.2 Quantum Entanglement

Quantum entanglement is a fundamental phenomenon in quantum mechanics where the quantum states of two or more particles become intrinsically correlated, regardless of the spatial separation between them. This correlation is so profound that the state of each particle cannot be fully described independently of the others. Entanglement challenges classical intuitions about locality and separability, reshaping our understanding of the quantum world. It also serves as a cornerstone in quantum information science, enabling groundbreaking applications such as quantum teleportation, superdense coding, and quantum cryptography.

2.2.1 Entangled States

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be the Hilbert spaces corresponding to two quantum systems, with bases $\{|0\rangle_1, |1\rangle_1\}$ and $\{|0\rangle_2, |1\rangle_2\}$, respectively. Consider a quantum state in the composite system $\mathbb{H}_1 \otimes \mathbb{H}_2$ given by:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_1 \otimes |0\rangle_2 + |1\rangle_1 \otimes |1\rangle_2\right). \tag{2.23}$$

This state exhibits a remarkable property: if a measurement on the first subsystem yields $|0\rangle_1$ (or $|1\rangle_1$), then the corresponding measurement on the second subsystem will yield $|0\rangle_2$ (or $|1\rangle_2$) with certainty. In other words, the measurement outcomes for $\mathbb{H}_1$ and $\mathbb{H}_2$ are perfectly correlated. This probabilistic dependence of measurement results is the hallmark of *quantum entanglement*. The state in equation 2.23 is known as *EPR state*, named after Einstein, Podolsky, and Rosen, who first discussed such correlations.

However, not all quantum states exhibit entanglement. Some states can be expressed as the tensor product of two vectors, one from each Hilbert space. For such states, measurements on one subsystem do not influence the other. Here is a modified version of the proof with enhanced clarity and conciseness:

Theorem 2.1. *Let $|\psi\rangle \in \bigotimes_{i=1}^n \mathbb{H}_i$. The results of measurement for each subsystem are independent if and only if $|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$ for some $|\phi_i\rangle \in \mathbb{H}_i$.*

Proof. ($\implies$) Let the basis for $\mathbb{H}_i$ be $\{|\varphi_1^{(i)}\rangle, \cdots, |\varphi_{m_i}^{(i)}\rangle\}$. Any state in $\mathbb{H}_i$ can be written as $\sum_{j_i=1}^{m_i} a_{j_i}^{(i)} |\varphi_{j_i}^{(i)}\rangle$, where $|a_{j_i}^{(i)}|^2$ represents the probability of obtaining $|\varphi_{j_i}^{(i)}\rangle$ upon measurement.

If $|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$, we can write:

$$|\psi\rangle = \left(\sum_{j_1=1}^{m_1} a_{j_1}^{(1)} |\varphi_{j_1}^{(1)}\rangle\right) \otimes \cdots \otimes \left(\sum_{j_n=1}^{m_n} a_{j_n}^{(n)} |\varphi_{j_n}^{(n)}\rangle\right). \tag{2.24}$$

Expanding this, we get:

$$|\psi\rangle = \sum_{j_1=1}^{m_1} \cdots \sum_{j_n=1}^{m_n} \left(a_{j_1}^{(1)} \cdots a_{j_n}^{(n)}\right) |\varphi_{j_1}^{(1)}\rangle \otimes \cdots \otimes |\varphi_{j_n}^{(n)}\rangle. \tag{2.25}$$

The probability of obtaining the outcome $|\varphi_{j_1}^{(1)}\rangle \otimes \cdots \otimes |\varphi_{j_n}^{(n)}\rangle$, that is, obtaining $|\varphi_{j_i}^{(i)}\rangle$ in the i^{th} subsystem for all $i = 1, \cdots, n$ is:

$$\text{Prob}(|\varphi_{j_1}^{(1)}\rangle, \cdots, |\varphi_{j_n}^{(n)}\rangle) = |a_{j_1}^{(1)} \cdots a_{j_n}^{(n)}|^2 = |a_{j_1}^{(1)}|^2 \cdots |a_{j_n}^{(n)}|^2. \tag{2.26}$$

The conditional probability of obtaining $|\varphi^{(i)}_{j_i}\rangle$ in the i^{th} subsystem, given the outcomes of the other subsystems, is:

$$\begin{aligned}
&\mathrm{Prob}(|\varphi^{(i)}_{j_i}\rangle \mid |\varphi^{(1)}_{j_1}\rangle, \cdots, \widehat{|\varphi^{(i)}_{j_i}\rangle}, \cdots, |\varphi^{(n)}_{j_n}\rangle) \\
&\quad = \frac{\mathrm{Prob}(|\varphi^{(1)}_{j_1}\rangle, \cdots, |\varphi^{(i)}_{j_i}\rangle, \cdots, |\varphi^{(n)}_{j_n}\rangle)}{\mathrm{Prob}(|\varphi^{(1)}_{j_1}\rangle, \cdots, \widehat{|\varphi^{(i)}_{j_i}\rangle}, \cdots, |\varphi^{(n)}_{j_n}\rangle)} \ .
\end{aligned} \tag{2.27}$$

Substituting the probabilities:

$$\mathrm{Prob}(|\varphi^{(i)}_{j_i}\rangle \mid \cdots) = \frac{|a^{(1)}_{j_1}|^2 \cdots |a^{(n)}_{j_n}|^2}{|a^{(1)}_{j_1}|^2 \cdots \widehat{|a^{(i)}_{j_i}|^2} \cdots |a^{(n)}_{j_n}|^2} = |a^{(i)}_{j_i}|^2 \ . \tag{2.28}$$

Since this is independent of the measurement outcomes of the other subsystems, the measurements are independent for separable states.

($\Longleftarrow$) Let $|\psi\rangle \in \bigotimes_{i=1}^n \mathbb{H}_i$. Suppose the measurement outcomes for each subsystem are independent. This means that for a measurement basis $\{|\varphi^{(i)}_{j_i}\rangle\}$ in each subsystem, the joint probability of obtaining the outcome $|\varphi^{(1)}_{j_1}\rangle \otimes \cdots \otimes |\varphi^{(n)}_{j_n}\rangle$ must satisfy:

$$\mathrm{Prob}(|\varphi^{(1)}_{j_1}\rangle, \cdots, |\varphi^{(n)}_{j_n}\rangle) = \prod_{i=1}^n \mathrm{Prob}(|\varphi^{(i)}_{j_i}\rangle) \ . \tag{2.29}$$

Write $|\psi\rangle$ in the product basis $\{|\varphi^{(1)}_{j_1}\rangle \otimes \cdots \otimes |\varphi^{(n)}_{j_n}\rangle\}$:

$$|\psi\rangle = \sum_{j_1,\ldots,j_n} c_{j_1,\ldots,j_n} |\varphi^{(1)}_{j_1}\rangle \otimes \cdots \otimes |\varphi^{(n)}_{j_n}\rangle \ . \tag{2.30}$$

The probability of obtaining the outcome $|\varphi^{(1)}_{j_1}\rangle \otimes \cdots \otimes |\varphi^{(n)}_{j_n}\rangle$ is:

$$\mathrm{Prob}(|\varphi^{(1)}_{j_1}\rangle, \cdots, |\varphi^{(n)}_{j_n}\rangle) = |c_{j_1,\ldots,j_n}|^2 \ . \tag{2.31}$$

From the independence assumption, we know:

$$|c_{j_1,\ldots,j_n}|^2 = p_1(j_1) \cdot p_2(j_2) \cdot \cdots \cdot p_n(j_n) \ , \tag{2.32}$$

where $p_i(j_i)$ depends only on j_i.

The condition $|c_{j_1,\ldots,j_n}|^2 = p_1(j_1) \cdot \cdots \cdot p_n(j_n)$ implies that the amplitudes $c_{j_1,\ldots,j_n}$ must factorize. That is:

$$c_{j_1,\ldots,j_n} = a^{(1)}_{j_1} a^{(2)}_{j_2} \cdots a^{(n)}_{j_n} \ , \tag{2.33}$$

where $a^{(i)}_{j_i}$ depends only on j_i.

Substitute the factorized form of $c_{j_1,\ldots,j_n}$ back into the expansion of $|\psi\rangle$:

$$|\psi\rangle = \sum_{j_1,\ldots,j_n} \left(a^{(1)}_{j_1} a^{(2)}_{j_2} \cdots a^{(n)}_{j_n} \right) |\varphi^{(1)}_{j_1}\rangle \otimes \cdots \otimes |\varphi^{(n)}_{j_n}\rangle \ . \tag{2.34}$$

Rearranging terms:

$$|\psi\rangle = \left(\sum_{j_1} a_{j_1}^{(1)} |\varphi_{j_1}^{(1)}\rangle\right) \otimes \left(\sum_{j_2} a_{j_2}^{(2)} |\varphi_{j_2}^{(2)}\rangle\right) \otimes \cdots \otimes \left(\sum_{j_n} a_{j_n}^{(n)} |\varphi_{j_n}^{(n)}\rangle\right) . \tag{2.35}$$

Define:

$$|\phi_i\rangle = \sum_{j_i} a_{j_i}^{(i)} |\varphi_{j_i}^{(i)}\rangle \in \mathbb{H}_i . \tag{2.36}$$

Thus, $|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$. □

From theorem 2.1, we can formalize the entanglement mathematically, we define separable and entangled states bases on algebra point of view as follows:

Definition 14 (Entangled and Separable States)**.** *Let $\mathbb{H}_1, \mathbb{H}_2, \ldots, \mathbb{H}_n$ be the Hilbert spaces associated with n quantum systems. A quantum state $|\psi\rangle \in \bigotimes_{i=1}^{n} \mathbb{H}_i$ is said to be* ***separable*** *if it can be written as the tensor product of n states:*

$$|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_n\rangle , \quad |\phi_i\rangle \in \mathbb{H}_i \ \forall i. \tag{2.37}$$

If such a decomposition is not possible, the state $|\psi\rangle$ is said to be ***entangled****.*

This distinction between separable and entangled states is fundamental to quantum mechanics and underpins many of its nonclassical features.

2.2.2 Classification

The entangled quantum states can be classified by SLOCC equivalent classes. SLOCC stands for *Stochastic Local Operations and Classical Communication* and is a concept primarily used in the context of quantum information theory and entanglement. The SLOCC equivalence is used to classify entangled states into equivalence classes, meaning that within the same class, states are "essentially" the same in terms of the type of entanglement they represent. Knowing whether states are SLOCC equivalent helps determine their usefulness in quantum protocols like quantum teleportation or superdense coding. Physically, two quantum states are considered SLOCC equivalent if one can be transformed into the other (and vice versa) using stochastic local operations and classical communication. In essence:

- Stochastic: The operations may succeed only probabilistically, meaning there is no guarantee of success in a single trial, but the transformation can theoretically occur with some non-zero probability.
- Local Operations (LO): Operations performed independently on each part of a multipartite system.
- Classical Communication (CC): Classical means allow communication between parties.

These conditions mean that by means of some basic quantum devices, these quantum states can be transformed to each other, which means that these quantum states can achieve the same quantum tasks through some simple operations. Technologically, these quantum states are viewed equivalent in quantum information.

Mathematically, the SLOCC equivalent relation is modeled by viewing the $\bigotimes_{i=1}^{n} SL(\mathbb{H}_i)$ as a group action on $\bigotimes_{i=1}^{n} \mathbb{H}_i$.

Definition 15 (SLOCC Equivalent). *Let $|\psi\rangle, |\varphi\rangle \in \bigotimes_{i=1}^{n} \mathbb{H}_i$ be two quantum states, they are called **SLOOC equivalent**, denoted by $|\psi\rangle \sim |\varphi\rangle$, if and only if there exists $A_i \in SL(\mathbb{H}_i)$, $i = 1, \cdots, n$ such that:*

$$|\psi\rangle = A_1 \otimes \cdots \otimes A_n |\varphi\rangle \ . \tag{2.38}$$

Example 2.4 (Bipartite system). *Consider the following two states in a two-qubit system with Hilbert space $\mathbb{H} \otimes \mathbb{H}$, where $\mathbb{H}$ having an orthonormal basis $\{|0\rangle, |1\rangle\}$:*

$$|\psi\rangle = |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \ , \quad |\varphi\rangle = \sqrt{2}\,|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \otimes |1\rangle \ . \tag{2.39}$$

Since there exist two matrices:

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \ , \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ , \tag{2.40}$$

where $A, B \in SL(\mathbb{H})$. Check that:

$$\begin{aligned} (A \otimes B)|\psi\rangle &= A|0\rangle \otimes B|0\rangle + A|1\rangle \otimes B|1\rangle \\ &= \sqrt{2}\,|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \otimes |1\rangle = |\varphi\rangle \ . \end{aligned} \tag{2.41}$$

Therefore, $|\psi\rangle \sim |\varphi\rangle$.

3 Algebraic Geometry of Quantum Mechanics

Quantum mechanics is a modern abstract theory still under development. Algebraic geometry, which encompasses both theoretical and computational aspects, is introduced to the study of quantum mechanics. In this chapter, we will discuss the most direct one, the projective geometry approach.

3.1 Projective Geometry and $U(1)$ Gauge Theory

We have discussed the quantum mechanics state in the previous section. The quantum states are equivalent in stationary quantum mechanics if $|\varphi\rangle = \mathrm{e}^{\mathrm{i}\,\theta}\,|\psi\rangle$. Physically, the stationary quantum mechanics is a $U(1)$ *gauge theory*. This reminds mathematicians of the projective space immediately! Recall that the underlying set of $\mathbb{P}^n_k$ is:

$$\mathbb{P}^n = \{[a_1 : \cdots : a_{n+1}] \mid [a_1, \cdots, a_{n+1}] \in k^{n+1} \backslash \{0\}\} \ , \tag{3.1}$$

where

$$\begin{aligned} &[a_1 : \cdots : a_{n+1}] = [b_1 : \cdots : b_{n+1}] \\ &\iff c[a_1, \cdots, a_{n+1}] = [b_1, \cdots, b_{n+1}] \text{ for some } c \in k^* \ . \end{aligned} \tag{3.2}$$

Let $n = \dim \mathbb{H} < \infty$, then we can describe the Hilbert space as a projective $n-$space over $\mathbb{C}$, we call this projective $(n-1)-$space *projective Hilbert space* and denote it by $\mathbb{P}(\mathbb{H})$.

Example 3.1. *In quantum information, the two-level systems, that is, the Hilbert spaces having dimension two, are often considered. Let* $\mathbb{H}$ *be a two-dimensional Hilbert space with basis* $\{|0\rangle, |1\rangle\}$*. The projective Hilbert space is:*

$$\mathbb{P}(\mathbb{H}) = \{[\psi_1 : \psi_2] \mid |\psi\rangle = (\psi_1, \psi_2) \in \mathbb{C}^2 \backslash \{0\}\} . \tag{3.3}$$

Since in $\mathbb{P}(\mathbb{H})$*,* $[\psi_1 : \psi_2] = A\,\mathrm{e}^{\mathrm{i}\,\theta}[\varphi_1 : \varphi_2]$*,* $\mathbb{P}(\mathbb{H})$ *is a two-dimensional, called* ***Bloch sphere*** *in quantum mechanics.*

This is the motivation for utilizing the projective geometry to describe the quantum systems. Now we want to consider the entanglement in the composite quantum systems, the tensor product is introduced now, so we need to introduce the notion of projective varieties here.

3.2 Varieties, Quantum States, and Entanglement

We have known that quantum entanglement is a fundamental feature of quantum mechanics, and can be classified and understood using algebraic geometry, particularly through the concept of secant and tangent varieties. These varieties provide a geometric framework to describe and distinguish different levels of entanglement in multipartite quantum systems.

3.2.1 Separable States for Distinguishable Particles as Segre Varieties

We start our discussion with variety descriptions of separable quantum states from the easiest distinguishable particle system. Recall that the simple tensor in the Hilbert spaces are separable quantum states:

$$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle \ . \tag{3.4}$$

From algebraic geometry, it is easy to see that the set of all such separable states has geometric meaning, which is given by the following theorem:

Theorem 3.1. *Let a* $n-$*distinguishable-particle system has the Hilbert space* $\mathbb{H} = \bigotimes_{i=1}^n \mathbb{H}_i$*. Then the set of all separable states forms a Segre variety:*

$$S \subseteq \mathbb{P}(\mathbb{H}) \ . \tag{3.5}$$

3.2.2 Coherent States for Identical Particles as Varieties

Just as we extended the concept of the Segre variety to the Veronese and Grassmann varieties, we can similarly extend the application of the Segre variety to describe separable states in quantum systems with symmetric or antisymmetric tensor product spaces. These quantum systems, characterized by such structures, are referred to as *identical particle systems.*

Separable States for Bosons as Veronese Varieties *Bosons* are a type of identical particle that obeys *Bose-Einstein statistics*, and their quantum states are symmetric under the exchange of particles. This symmetry means that the overall quantum state remains unchanged if one swaps any two bosons in the system. Hence, intuitively, instead of regarding the Hilbert space of a bosonic composite system as a tensor product of Hilbert spaces $\bigotimes_{i=1}^n \mathbb{H}$, we regard it as the symmetric tensor of Hilbert spaces product $\mathrm{Sym}^n\,\mathbb{H}$.

As we previously discussed, in quantum mechanics, a *separable state* for a composite system can be written as a product of states from its constituent subsystems. The notion of separable states changes to *coherent state* for composite systems of bosons.

Definition 16. *A quantum state* $|\varphi\rangle \in \mathrm{Sym}^n \mathbb{H}$ *is called a* ***coherent state*** *if and only if there exists* $|\psi\rangle = \sum_{j=1}^{d} |j\rangle \in \mathbb{H}$ *such that:*

$$|\varphi\rangle = |\psi\rangle^{\odot n} \ . \tag{3.6}$$

Hence, naturally, we can consider the following map:

$$\Omega : \mathbb{P}(\mathbb{H}) \cong \mathbb{P}^{d-1}_{\mathbb{C}} \to \mathbb{P}(\mathrm{Sym}^n \mathbb{H}) \ , \quad [|\psi\rangle] \mapsto [|\varphi\rangle] \ , \tag{3.7}$$

which is indeed a Veronese embedding and satisfies:

$$\mathrm{Im}\, \Omega = V \subseteq \mathbb{P}(\mathrm{Sym}^n \mathbb{H}) \ . \tag{3.8}$$

From the above argument, the set of all such coherent states has geometric meaning:

Theorem 3.2. *Let a* $n-$*identical-boson system has the Hilbert space* $\mathrm{Sym}^n\mathbb{H}$*. Then the set of all separable states forms the Veronese variety:*

$$V \subseteq \mathbb{P}\left(\mathrm{Sym}^n \mathbb{H}\right) \ . \tag{3.9}$$

Hence, the Veronese variety V parametrizes the coherent states for n bosons, capturing the geometry of symmetric quantum states. These states are inherently simpler to describe compared to entangled states.

Separable States for Fermions as Grassmann Varieties Fermions, in contrast to bosons, obey *Fermi-Dirac statistics*, which require their quantum states to be antisymmetric under particle exchange. This antisymmetry means that the overall quantum state has an additional -1 factor if one swaps any two fermions in the system. In addition. The antisymmetry also leads to the well-known *Pauli exclusion principle*, stating that no two fermions can occupy the same quantum state.

Like bosons, the notion of separable states changes to coherent states as well. For n fermions, coherent states are constructed using the wedge product and the Hilbert space of a fermionic composite system is the wedge product of Hilbert spaces $\bigwedge^n \mathbb{H}$. Just like previous discussion, the coherent states can be understood geometrically.

Theorem 3.3. *Let a* $n-$*identical-fermion system has the Hilbert space* $\bigwedge^n \mathbb{H}$*. Then the set of all separable states forms the Grassmann variety:*

$$G \subseteq \mathbb{P}\left(\bigwedge\nolimits^n \mathbb{H}\right) \ . \tag{3.10}$$

Due to the antisymmetric nature of fermionic systems, they are inherently more complex to analyze than bosonic systems. While the proof of this theorem follows a similar structure to the bosonic case, it requires additional physical arguments specific to fermions. For brevity, we will omit the detailed proof here.

The Grassmann variety G parametrizes the separable states for n fermions. It represents the geometric space of antisymmetric quantum states, reflecting the constraints imposed by Fermi-Dirac statistics.

Both Veronese and Grassmann varieties provide elegant geometric descriptions of the coherent states for bosonic and fermionic systems, respectively. These varieties also serve as a foundation for understanding more complex quantum states, such as entangled states.

3.2.3 Entangled States as Secant and Tangent Varieties

The secant varieties of the Segre variety, Veronese variety, or Grassmann variety generalize the notion of separability or coherent states and provide a geometric hierarchy for classifying quantum states based on their entanglement properties. We have discussed that the $k-$secant variety $\sigma_k(X)$ of the Segre variety, Veronese variety, or Grassmann variety, denoted by X is the Zariski closure of the union of projective spaces spanned by points on X. Physically, $\sigma_k(X)$ corresponds to quantum states that can be expressed as superpositions of less than or equal to k separable or coherent states.

Moreover, since we have theorem 1.1, an inclusion sequence of secant varieties. Consider the sequence for secant varieties of the variety X, this hierarchy of secant varieties provides a geometric measure of the degree of entanglement. That is to say, let $\mathbb{H}$ be the Hilbert space for a distinguishable particle, fermionic, or bosonic quantum system, that has one of the following forms:

$$\bigotimes_{i=1}^{n} \mathbb{H}_i \ , \quad \mathrm{Sym}^n \, \mathbb{H}_1 \ , \quad \bigwedge^n \mathbb{H}_1 \ . \tag{3.11}$$

We can obtain an inclusion sequence:

$$X = \sigma_1(X) \subseteq \sigma_2(X) \subseteq \sigma_3(X) \subseteq \cdots \subseteq \mathbb{P}(\mathbb{H}) \ . \tag{3.12}$$

States in the proper $k-$secant varieties $\varsigma_k(X) = \sigma_k(X) \backslash \sigma_{k-1}(X)$ with larger k value require more separable or coherent states to be expressed. Hence, larger values for the number k indicate higher levels of entanglement.

Moreover, we can introduce the tangent varieties of X, the $k-$tangent varieties are contained in the $k-$secant variety:

$$\tau_k(X) \subseteq \sigma_k(X) \ . \tag{3.13}$$

The tangent varieties also represent parts of the entangled states of the quantum systems. We will see more applications of representing entangled states as tangent and secant varieties in the next section.

4 Applications: Entangled States Classification

So far, we have discussed the relations between the notions in quantum mechanics and algebraic geometry, which is summarized in the table 1.

Since we know how to utilize algebraic geometry to describe the notions in quantum mechanics, we can see some applications to understand why algebra can be useful in quantum mechanics.

Since the entangled states can be described by varieties, we can apply the varieties to classify the entangled states with different degrees of entanglement. Before applying First we need the following lemma:

Lemma 4.1. *The Segre variety, Veronese variety, and Grassmann variety are SLOCC invariant.*

Quantum mechanics	Algebraic geometry
Quantum Hilbert space	Projective space
Entanglement classes	Projective varieties
Separable states for distinguishable particles	Segre varieties
Separable states for bosons	Veronese varieties
Separable states for fermions	Grassmann varieties
Entangled states	Secant and tangent varieties
Degree of entanglement	Indices of proper secant varieties

Table 1: Relations between quantum mechanics and algebraic geometry.

Proof. We prove the case for the Segre variety. Consider a Segre embedding:

$$\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_k) \hookrightarrow \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_k) \ , \quad ([v_1], \cdots, [v_n]) \mapsto [v_1 \otimes \cdots \otimes v_n] \ . \tag{4.1}$$

Let $G = GL(V_1) \times GL(V_2) \times \cdots \times GL(V_k)$ and V_i be vector spaces over a field K (say $\mathbb{C}$), and $\dim(V_i) = n_i$. G acts on $V_1 \times V_2 \times \cdots \times V_k$ by:

$$(g_1, g_2, \ldots, g_k) \cdot (v_1, v_2, \ldots, v_k) = (g_1 v_1, g_2 v_2, \ldots, g_k v_k) \ , \tag{4.2}$$

where $g_i \in GL(V_i)$ and $v_i \in V_i$.

On $\mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_k)$, the group G acts naturally by:

$$(g_1, g_2, \ldots, g_k) \cdot [w] = [(g_1 \otimes g_2 \otimes \cdots \otimes g_k) w] \ , \tag{4.3}$$

for $w \in V_1 \otimes V_2 \otimes \cdots \otimes V_k$ and $g_i \in GL(V_i)$. We now check that:

$$\sigma((g_1, g_2, \ldots, g_k) \cdot ([v_1], [v_2], \ldots, [v_k])) = (g_1, g_2, \ldots, g_k) \cdot \sigma([v_1], [v_2], \ldots, [v_k]) \ . \tag{4.4}$$

The left-hand side is:

$$\begin{aligned} \sigma\big((g_1, g_2, \ldots, g_k) \cdot ([v_1], [v_2], \ldots, [v_k])\big) &= \sigma([g_1 v_1], [g_2 v_2], \ldots, [g_k v_k]) \\ &= [g_1 v_1 \otimes g_2 v_2 \otimes \cdots \otimes g_k v_k] \ . \end{aligned} \tag{4.5}$$

The right-hand side is:

$$\begin{aligned} (g_1, g_2, \ldots, g_k) \cdot \sigma([v_1], [v_2], \ldots, [v_k]) &= (g_1, g_2, \ldots, g_k) \cdot [v_1 \otimes v_2 \otimes \cdots \otimes v_k] \\ &= [(g_1 \otimes g_2 \otimes \cdots \otimes g_k)(v_1 \otimes v_2 \otimes \cdots \otimes v_k)] \\ &= [g_1 v_1 \otimes g_2 v_2 \otimes \cdots \otimes g_k v_k]. \end{aligned} \tag{4.6}$$

Hence, the Segre embedding is G-equivariant. Since the orbits for $GL(V_i)$ acting on V_i are $\{0\}$ and $V_i \backslash \{0\}$ and $PGL(V_i) \cong PSL(V_i)$ for V_i over $\mathbb{C}$, Segre variety is a SLOCC orbit and therefore SLOCC invariant. □

Lemma 4.2. *The $k-$tangent and $k-$secant varieties of Segre variety, Veronese variety, and Grassmann variety are SLOCC invariant.*

Proof. Since the corresponding variety X is SLOCC invariant, it is easy to see that the $k-$secant variety which is spanned from X is SLOCC invariant as well. □

We can now apply these lemmas to classify the entangled states. Since we know that the separable states or coherent states form certain Segre, Veronese, or Grassmann varieties X and by lemma 4.1 and 4.2, the varieties in the inclusion sequence given in theorem 1.1 and $\tau_k(X) \subseteq \sigma_k(X)$, which are partial order relations, are all SLOCC invariant, we can propose the following theorem for classification of entangled states:

Theorem 4.3 (Entangled states classification)**.** *Given a Hilbert space for a quantum system, we can classify the quantum states in the Hilbert space into several classes by the following steps:*

- *Identify the Segre variety, Veronese variety, or the Grassmann variety of the projective Hilbert space $\mathbb{P}(\mathbb{H})$. Denote it as X.*
- *Classify the entangled states by identifying the sequence of $k-$secant variety:*

$$X = \sigma_1(X) \subseteq \sigma_2(X) \subseteq \cdots \sigma_n(X) \subseteq \mathbb{P}(X) \ . \tag{4.7}$$

  If $\varsigma_k(X) \neq \varnothing$, then each $\varsigma_k(X)$ represents an entangled state family.

- *Identify the $k-$tangent variety for each $\sigma_k(X)$:*

$$\tau_k(X) \subseteq \sigma_k(X) \ . \tag{4.8}$$

  If $\tau_k(X) \subsetneqq \sigma_k(X)$, then $\tau_k(X)$ $\varsigma_k(X)\backslash\tau_k(X)$ represent two different entangled state families.

- *Make a table for the entangled states from the above information:*

Entangled state families	Varieties
Class N	$\tau_k(X)$
Class $N-1$	$\varsigma_k(X)\backslash\tau_k(X)$
$\vdots$	$\vdots$
Class 1	X

Remark 8. *The theorem 4.3 can not always provide all SLOCC classes, especially for high-dimensional spaces, and therefore called families here. We need to identify the number of SLOCC classes in each projective Hilbert space to make sure that we obtain all classes or not. If we have not obtained all classes, then we can apply other methods, such as ranks, to classify more precisely. These methods may not be related to algebraic geometry and therefore will not discuss it here. See the references for more details about it.*

From the table obtained in theorem 4.3, we can make an onion-like entangled state family diagram as Figure 1.

4.1 Distinguishable Particles Entangled States Classification

Since the Segre variety is the easiest variety for the separable or coherent states, We first apply theorem 4.3 to the easy distinguishable two-qubit system, which is well-studied by Schmidt decomposition, as an easy example.

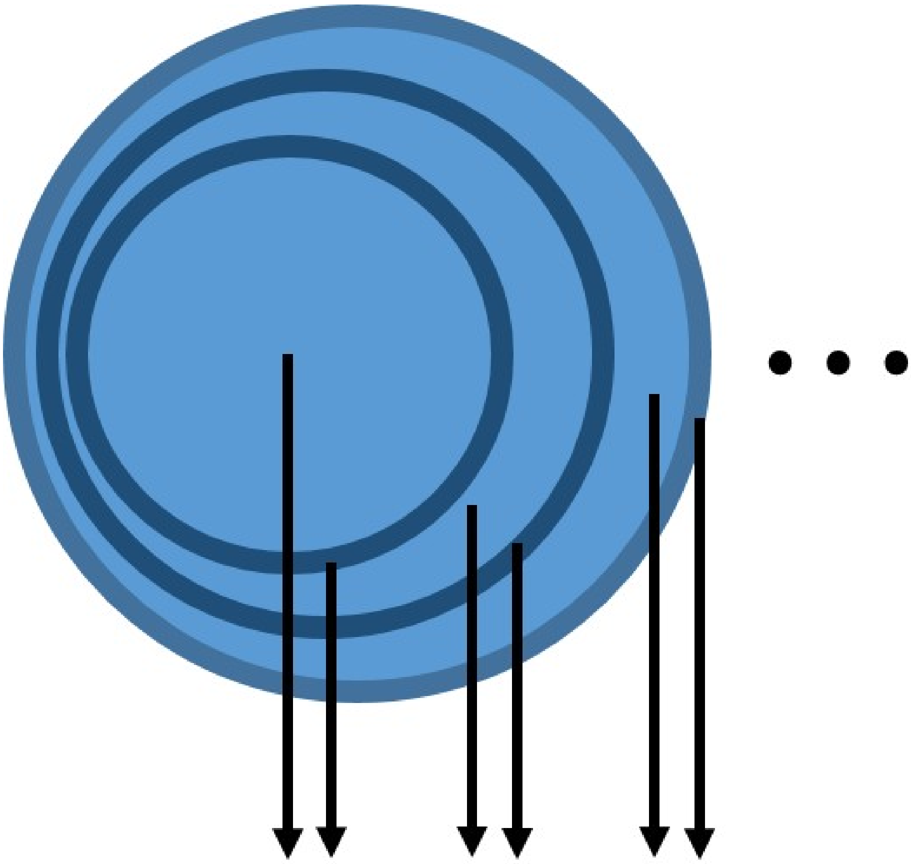

Figure 1: Entangled state family diagram.

Example 4.1 (Distinguishable two-qubit system). *Consider a two-qubit quantum system with Hilbert space with Hilbert space $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_1 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$, where $\mathbb{H}_1$ having orthonormal basis $\{|0\rangle, |1\rangle\}$. The projective Hilbert space is $\mathbb{P}(\mathbb{H}) \cong \mathbb{P}^3_{\mathbb{C}}$. The Segre variety for $\mathbb{P}^3_{\mathbb{C}}$ is given by:*

$$X = \left\{ \left[1 : a : b : ab \right] \in \mathbb{P}(\mathbb{H}) \;\middle|\; a, b \in \mathbb{C} \right\} . \tag{4.9}$$

We apply the proper secant variety first. The 2−secant variety of X is given by:

$$\sigma_2(X) = \left\{ \left[\lambda_1 + \lambda_2 : \lambda_1 a_1 + \lambda_2 a_2 : \lambda_1 b_1 + \lambda_2 b_2 : \lambda_1 a_1 b_1 + \lambda_2 a_2 b_2 \right] \in \mathbb{P}(\mathbb{H}) \;\middle|\; \lambda_1, \lambda_2, a, b \in \mathbb{C} \right\} . \tag{4.10}$$

Consider the following states:

$$[1:0:0:1] = [1:1:1:1] + [1:-1:-1:1] , \tag{4.11}$$
$$[1:0:0:-1] = [1:1:-1:-1] + [1:-1:1:-1] , \tag{4.12}$$
$$[0:1:1:0] = [1:1:1:1] - [1:-1:-1:1] , \tag{4.13}$$
$$[0:1:-1:0] = [1:1:-1:-1] - [1:-1:1:-1] , \tag{4.14}$$

*which are easy to see in $\sigma_2(X)$. In $\mathbb{P}(\mathbb{H})$ the corresponding four states are called entangled **Bell states** or **Bell basis**, which represent the entangled states in $\mathbb{P}(\mathbb{H})$. Therefore, the $2-$secant variety fills all $\mathbb{P}(\mathbb{H}) \cong \mathbb{P}^3_{\mathbb{C}}$. We can conclude the sequence:*

$$X = \sigma_1(X) \subseteq \sigma_2(X) = \mathbb{P}^3_{\mathbb{C}} \cong \mathbb{P}(\mathbb{H}) \; . \tag{4.15}$$

We now move to the tangent variety. The $2-$tangent star variety of X is given by:

$$\tau_2(X) = \bigcup_{p \in X} T^*_p(X) \; , \tag{4.16}$$

where the tangent star at $p = [1 : a : b : ab] \in X$ can be calculated as follows:

$$\begin{aligned} T^*_p(X) &= \left\{ \lim_{\epsilon \to 0} \frac{\lambda}{\epsilon}([1 : a+\epsilon : b+\epsilon : (a+\epsilon)(b+\epsilon)] - [1 : a : b : ab]) + [1 : a : b : ab] \;\middle|\; \lambda \in \mathbb{C} \right\} \\ &= \{\lambda[0 : 1 : 1 : a+b] + [1 : a : b : ab] \mid \lambda \in \mathbb{C}\} \; . \end{aligned} \tag{4.17}$$

Therefore, we can conclude that:

$$\tau_2(X) = \bigcup_{p \in X} T^*_p(X) = \mathbb{P}^3_{\mathbb{C}} = \sigma_2(X) \cong \mathbb{P}(\mathbb{H}) \; . \tag{4.18}$$

So we can make the following table: This is the complete result for the distinguishable two-

Names	SLOCC classes	Varieties
Bell states	$\|0\rangle \otimes \|0\rangle + \|1\rangle \otimes \|1\rangle$	$\varsigma_2(X)$
Separable states	$\|0\rangle \otimes \|0\rangle$	X

Table 2: Table for the SLOCC classes, their corresponding varieties.

qubit system, which satisfies the known result obtained from Schmidt decomposition.

We can see that the Bell states we obtained from the $\varsigma_2(X)$ in the inclusion sequence $X = \sigma_1(X) \subseteq \sigma_2(X) = \mathbb{P}^3_{\mathbb{C}} \cong \mathbb{P}(\mathbb{H})$, is indeed more entangled than the separable states in $\sigma_1(X) = X$.

4.2 Bosonic Entangled States Classification

To classify the bosonic systems by applying the Veronese variety, we need to describe the Veronese variety and its projective space first. Let $\{|j\rangle\}_{i=1}^d$ be an orthonormal basis for $\mathbb{H}$. For a composite bosonic system, for all coherent states $|\varphi\rangle \in \mathrm{Sym}^n \mathbb{H}$, there exists $|\psi\rangle = \sum_{j=1}^d |j\rangle \in \mathbb{H}$ such that:

$$|\varphi\rangle = |\psi\rangle^{\odot n} \; . \tag{4.19}$$

Therefore, we can write:

$$|\varphi\rangle = \sum_{n_1+\cdots+n_d=n} \prod_{j=1}^d x_j^{n_j} \sum_{\alpha \in F[n_1,\cdots,n_d]} |\alpha(1)\rangle \otimes \cdots \otimes |\alpha(n)\rangle \; , \tag{4.20}$$

where the set $F[n_0, \cdots, n_d]$ is defined as:

$$F[n_0, \cdots, n_d] = \left\{ \alpha : \{1, \cdots, n\} \to \{1, \cdots, d\} \mid \alpha \text{ has value } j \text{ for } n_j \text{ times} \right\} . \tag{4.21}$$

Then we can represent $[|\varphi\rangle] \in \mathbb{P}(\operatorname{Sym}^n \mathbb{H})$ and $[|\psi\rangle] \in \mathbb{P}(\mathbb{H})$ in homogeneous coordinates:

$$[|\psi\rangle] = [x_1, \cdots, x_d] , \quad [|\varphi\rangle] = [x_1^n : x_1^{n-1}x_2 : x_1^{n-1}x_3 : \cdots : x_{d-1}x_d^{n-1} : x_d^n] . \tag{4.22}$$

Example 4.2 (Bosonic three-qubit system)**.** *Consider a three-qubit quantum system with Hilbert space $\mathbb{H} = \operatorname{Sym}^3 \mathbb{H}$, where $\mathbb{H}$ having an orthonormal basis $\{|0\rangle, |1\rangle\}$. The Veronese variety is therefore given by:*

$$V = \left\{ \left[1 : x : x^2 : x^3 \right] \,\middle|\, x \in \mathbb{C} \right\} . \tag{4.23}$$

The 2−secant variety is given by:

$$\sigma_2(V) = \left\{ \left[\lambda_1 + \lambda_2 : \lambda_1 x + \lambda_2 y : \lambda_1 x^2 + \lambda_2 y^2 : \lambda_1 x^3 + \lambda_2 y^3 \right] \,\middle|\, \lambda_i, x, y \in \mathbb{C} \right\} . \tag{4.24}$$

We can see that $\sigma_2(X) = \mathbb{P}(\operatorname{Sym}^3 \mathbb{H}) \cong \mathbb{P}^3_{\mathbb{C}}$. Hence, we can conclude the sequence:

$$V = \sigma_1(V) \subseteq \sigma_2(V) = \mathbb{P}(\operatorname{Sym}^3 \mathbb{H}) . \tag{4.25}$$

Hence, we can conclude The 2−tangent variety can be calculated by considering a point $p = [1 : x : x^2 : x^3] \in V$. The tangent for p is:

$$\begin{aligned} T_p^*(V) &= \left\{ \lim_{\epsilon \to 0} \frac{\lambda}{\epsilon} \left([1 : (x+\epsilon) : (x+\epsilon)^2 : (x+\epsilon)^3] - [1 : x : x^2 : x^3] \right) + p \,\middle|\, \lambda \in \mathbb{C} \right\} \\ &= \{ [2 : \lambda + x : 2\lambda x + x^2 : 3\lambda x^2 + x^3] \mid \lambda \in \mathbb{C} \} . \end{aligned} \tag{4.26}$$

Hence, the tangent star variety is:

$$\tau_2(V) = \bigcup_{p \in V} T_p^*(V) = \{ [2 : \lambda + x : 2\lambda x + x^2 : 3\lambda x^2 + x^3] \mid x, \lambda \in \mathbb{C} \} . \tag{4.27}$$

It is easy to see that $\tau_2(V) \subsetneq \sigma_2(V)$, so we can make the following table:

Names	SLOCC classes	Varieties
W states	$\|0\rangle \odot \|1\rangle \odot \|1\rangle$	$\tau_2(V)$
GHZ states	$\|0\rangle \odot \|0\rangle \odot \|0\rangle + \|1\rangle \odot \|1\rangle \odot \|1\rangle$	$\varsigma_2(V) \backslash \tau_2(V)$
Separable states	$\|0\rangle \odot \|0\rangle \odot \|0\rangle$	V

Table 3: Table for the SLOCC classes, their corresponding varieties.

This is the complete SLOCC class for the system, satisfying the known result for a three-qubit bosonic system.

The W states belong to the tangent star variety $\tau_2(V)$ consists of points spanned by the limits of secant lines. This geometry indicates that W states are not maximally entangled but still possess multipartite entanglement. The tangent geometry implies robustness to particle *loss, making W states ideal for quantum networks and communication where qubit loss is a concern.*

GHZ states are part of the second secant variety $\sigma_2(V)$*, excluding the tangent star variety. This means GHZ states represent a higher-dimensional subvariety with maximal multipartite entanglement. The broader geometric space and high-dimensional entanglement structure make GHZ states perfect for applications requiring maximal correlations, such as quantum nonlocality and fault-tolerant quantum computing.*

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