Green Functions, Generalized Functions, and Their Applications in Physics

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Preface

Modern theoretical mechanics constantly appeals to objects such as the Dirac delta, impulse responses, and Green functions. In many standard courses these appear as formal tools; the goal of this supplement is to put them on solid ground while keeping the physics front and center. We develop just enough analysis—metric and topological preliminaries, measures and distributions, Fourier analysis and residues—to treat Green operators for common ODE and PDE models (oscillator, Poisson, Helmholtz, and wave equations) in a way that is rigorous, concise, and useful for calculations.

This is **not** a full text in analysis or PDE. It is a pragmatic bridge for students of mechanics who want a coherent path from physical motivation to mathematically sound statements. Throughout we emphasize:

- modeling assumptions and domains of validity,
- causality and retarded solutions,
- convolution algebra and impulse responses,
- distributional derivatives, and
- the interplay between time-domain intuition and frequency-domain techniques.

Scope and Learning Outcomes

After working through these notes, a reader should be able to:

- formulate linear time-invariant systems and field equations using distributions;
- compute and interpret Green functions for standard ODE and PDE operators with physical boundary or initial data;
- pass between time and frequency domains using the Fourier transform and residue calculus in a controlled way;
- justify common manipulations with δ, Θ, and principal-value distributions;
- read more advanced texts on PDEs and field theory with less friction.

Prerequisites (Minimal)

Multivariable calculus, linear algebra, and an introductory ODE course. Helpful but not required: basic complex analysis (Cauchy integral theorem), familiarity with inner products and norms, and a first look at L^p spaces. Key reminders are provided inline or as short appendices.

How to Use These Notes

First pass (physics-forward): Skim Section 1 (metric and topology essentials) for vocabulary; read Section 2 on measures and distributions with examples; then tackle Sections 3–4 on Fourier transforms, residues and Green operators; finally return to any background you found unfamiliar.

Second pass (rigor upgrade): Revisit the measure and distribution definitions and derivations, and work through the contour-integral derivation of a representative Green function step by step.

Each major section ends with short "checkpoints" highlighting common pitfalls (e.g., multiplying distributions indiscriminately, confusing pointwise with weak convergence).

Notation and Conventions

- Spaces: $\mathbb{R}^n$ with the Euclidean topology; $\mathcal{S}$ the Schwartz space; $\mathcal{S}'$ tempered distributions; C_c^∞ smooth compactly supported functions.
- Operators: convolution $f * g$; distributional derivative ∂^α; pushforward of a measure/distribution by a map Φ is $\Phi_{\#}\mu$.
- Special distributions: Dirac δ, Heaviside Θ, principal value $\mathrm{p.\,v.}(1/x)$.
- Fourier transform: $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi \mathrm{i} x\cdot\xi}, dx$ (this sign convention is used consistently unless stated otherwise).
- Topology: closure $\overline{A}$, interior $\mathrm{int}(A)$, and boundary $\partial A = \overline{A} \setminus \mathrm{int}(A)$.
- Flows: we denote a flow on a state space X by $\mathbf{\Phi}_t : X \to X$ (upper-case Φ is used consistently).

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Chapter 1

Mathematical Background

1.1 Metric Spaces

We must first make precise what it means for two initial conditions to be "close." In physical systems, this idea is intuitive: two pendulums starting from nearly identical positions swing almost together at first but may soon diverge dramatically. This sensitivity to initial conditions is one of the hallmarks of chaotic behavior.

However, to **quantify** this closeness and divergence, we need a mathematical notion of distance on the state space. Without such a structure, phrases like "arbitrarily close initial points" or "trajectories that eventually separate" are meaningless. The appropriate formal tool is a **metric**, a function that assigns to every pair of states a nonnegative real number representing their distance.

Introducing a metric allows us to translate intuitive physical notions—such as nearby positions or similar velocities—into precise mathematical statements.

Definition 1.1: Metric

A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ (symmetry).
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

The pair (X, d) is called a **metric space**.

Example 1.1

- **Euclidean metric:** the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x, y) = |x - y|$ is certainly a metric.
- **Discrete metric:** for any set X, there always exists a metric called **discrete metric** defined as $d : X \times X \to \mathbb{R}$,

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y \end{cases}$$

- **Norm:** the norms defined in inner product spaces are metric.

Once a metric d is defined on the state space X, we can describe:

- what it means for two trajectories to approach or diverge;
- continuity of the flow φ_t with respect to initial conditions;
- and, ultimately, the notion of **sensitive dependence on initial conditions**.

In this way, the metric serves as the bridge between geometry and dynamics. It allows us to measure how trajectories evolve relative to one another, making possible a quantitative formulation of chaos. The following section introduces the concept of a metric space, which provides this essential structure.

1.1.1 Open and Closed Sets

Once a metric is defined, we can speak not only about the distance between two points but also about the **neighborhood** of a point—those states that lie within a certain distance from it. This idea is fundamental for describing continuity, stability, and the local behavior of trajectories in a dynamical system.

In physics, we often reason about "nearby configurations" or "small perturbations" of a state. For instance, when we say a system is stable under small disturbances, we mean that if we slightly change the initial condition, the resulting trajectory stays within a nearby region. Mathematically, such a "nearby region" is captured by an **open ball** around a point in the metric space.

Definition 1.2: Open ball

Let (X, d) be a metric space. For $x_0 \in X$ and $r > 0$, the **open ball** of radius r centered at x_0 is defined as

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}.$$

The collection of all open balls leads us to the concept of **open sets**, which formalize the intuitive notion of regions without boundary points included.

Definition 1.3: Open set

Let (X, d) be a metric space. A set $U \subseteq X$ is said to be **open** if for every $x \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$.

The complements of open sets, the **closed sets**, represent regions that contain their boundaries—important when studying invariant sets or attractors that trap trajectories over time.

Definition 1.4: Closed set

Let (X, d) be a metric space. A set $C \subseteq X$ is said to be **closed** if its complement $X \setminus C$ is open.

Example 1.2

- The interval $(a, b) \subseteq \mathbb{R}$ is certainly open under the Euclidean metric.

  Proof. For every $x \in (a, b)$, there always exists $r = \min\{|x-a|, |x-b|\}/2$ so that $B_r(x) = (x - r, x + r) \subseteq (a, b)$. □

- All sets are open under the discrete metric.

  Proof. Let (X, d) be a metric space where d is the discrete metric and $U \subseteq X$. For any $x \in U$, there always exists $r = 1/2$ so that $B_r(x) = \{x\} \subseteq U$. □

1.1.2 Interior, Closure, and Boundary

With the notions of neighborhoods and open sets, we can now describe how a set occupies space within a metric space. In physical terms, this corresponds to distinguishing the **bulk** of a region—where the system can move freely without approaching the edge—from its **boundary**, where the behavior changes qualitatively.

Definition 1.5: Interior

Let (X, d) be a metric space and $A \subseteq X$. The **interior** of $A \subseteq X$ is

$$\operatorname{int}(A) = \{x \in A : \exists r > 0, B_r(x) \subseteq A\}.$$

Definition 1.6: Boundary

Let (X, d) be a metric space and $A \subseteq X$. The **boundary** of A is

$$\partial A = \overline{A} \setminus \operatorname{int}(A).$$

For example, in phase space, a region of stable motion (such as an island of regular trajectories in a mixed system) has an interior where trajectories remain confined, while its boundary separates it from chaotic zones. The closure of a set includes all points that can be approached by states within it, even if they are not themselves inside the region—just as the edge of a potential well may be reached asymptotically by nearby trajectories.

Definition 1.7: Closure

Let (X, d) be a metric space and $A \subseteq X$. The **closure** of A is

$$\overline{A} = \{x \in X : \text{for all } r > 0, B_r(x) \cap A \neq \emptyset\}.$$

These geometric ideas are not merely abstract. When we study chaotic systems, we often encounter sets whose closure fills the entire phase space—meaning their trajectories come arbitrarily close to every possible state. Such sets are said to be **dense**, and they play a central role in the mathematical definition of chaos.

Definition 1.8: Dense Set

Let (X, d) be a metric space and $A \subseteq X$. A subset $A \subseteq X$ is **dense** in X if every point $x \in X$ is either in A or arbitrarily close to a point of A:

$$\overline{A} = X.$$

Example 1.3

- Rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$.
- Periodic points can be dense in a chaotic map (e.g. logistic map).

1.1.3 Continuity and Neighborhoods

The notions of open and closed sets allow us to describe how points and trajectories behave locally within a metric space. In dynamical systems, this local structure captures how small perturbations of an initial state affect its future evolution.

Definition 1.9: Neighborhood

Let (X, d) be a metric space and $x_0 \in X$. A set $U \subseteq X$ is called a **neighborhood** of x_0 if there exists $r > 0$ such that the open ball $B_r(x_0)$ is contained in U. In other words, U contains an entire region around x_0.

Definition 1.10: Continuous map

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is said to be **continuous at a point** $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon.$$

The map f is **continuous** if it is continuous at every point $x \in X$.

Remark 1.1: Physical interpretation

Continuity expresses the idea that small changes in the initial state produce small changes in the outcome. In mechanics, this corresponds to systems where slightly different initial positions or velocities lead to nearby trajectories—at least for short times. When this property fails, the system exhibits extreme sensitivity to initial conditions, one of the signatures of chaos.

Example 1.4: Continuity of flows

If (X, Φ_t) is a dynamical system on a metric space, we usually assume that for each fixed t, the map $\Phi_t : X \to X$ is continuous. This ensures that the

flow evolves states smoothly over time. For instance, in classical mechanics, the flow generated by a smooth vector field $F(x)$ is continuous with respect to initial data.

The combination of metric and topological concepts—open sets, neighborhoods, and continuity—provides the language for expressing local stability and sensitivity. Later, when we define chaotic systems, these notions will allow us to state precisely what it means for trajectories that start arbitrarily close to eventually separate by a definite distance.

1.2 Measure Theory and Generalized Functions

Having defined a notion of distance and continuity on the state space, we now turn to the question of **quantifying** how much of the space is occupied by certain sets or trajectories. In physics, this corresponds to measuring quantities such as total mass, charge, or probability — objects that may be distributed continuously or concentrated at single points. To describe both continuous and discrete distributions within one unified framework, we require the language of **measure theory**.

Moreover, some physical quantities—like the Dirac delta appearing in point charges or impulse forces—cannot be represented as ordinary functions. This motivates the introduction of generalized functions (or distributions), which extends the notion of function to include such singular objects.

1.2.1 Motivation

In classical mechanics, we usually describe motion by specifying the trajectory in real space

$$\mathbf{x}(t) \in \mathbb{R}^n$$

of a particle as it evolves under the equation of motion

$$\dot{\mathbf{x}} = f(\mathbf{x}),$$

or, in Hamiltonian form,

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}.$$

Such a description is perfectly adequate when we can follow a single trajectory in detail. However, many situations in physics—from statistical mechanics to chaotic dynamics— require us to describe not just one trajectory but an entire **ensemble** of possible states.

To move from individual trajectories to an ensemble description, we introduce a density function

$$\rho(\mathbf{x}, t) \in (0, \infty) \text{ and } \int_{\mathbb{R}^n} \rho(\mathbf{x}, t) \mathrm{d}^n\mathbf{x} = 1$$

which tells us how the probability or "weight" of states is distributed in phase space. Its time evolution is governed by the Liouville equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (f\rho) = 0,$$

which expresses the local conservation of probability along the phase flow.

1.2.2 Dirac Measure as a Singular Distribution

When the system is known to be **exactly at some point** $\mathbf{x}_0 \in \mathbb{R}^n$ **at time** $t = 0$, the corresponding initial density is not a smooth function but a sharply

concentrated **measure**:

$$\rho(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0),$$

the so-called **Dirac delta measure**. It is defined by its action on test functions $\phi(x)$:

$$\int_{\mathbb{R}^n} \phi(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{x}_0)\, \mathrm{d}^n\mathbf{x} = \phi(\mathbf{x}_0).$$

Thus, the delta function should not be viewed as an ordinary function, but rather as a measure that assigns unit weight to the single point $\mathbf{x}_0$.

Under the deterministic flow $\Phi_t(\mathbf{x}_0)$ generated by $\dot{\mathbf{x}} = f(\mathbf{x})$, this measure is simply **pushed forward**:

$$\rho(\mathbf{x}, t) = \delta\big(\mathbf{x} - \Phi_t(\mathbf{x}_0)\big).$$

In this way, the Dirac measure represents the evolution of a single trajectory within the general measure-theoretic framework.

The same idea reappears in the study of linear dynamical systems. Consider, for instance, the damped harmonic oscillator,

$$m\ddot{x} + \gamma\dot{x} + kx = f(t).$$

If we apply a unit impulse at $t = 0$,

$$f(t) = \delta(t),$$

the resulting motion $x(t) = G(t)$ defines the **Green function** (or **impulse response**) of the system. For an arbitrary external force, the displacement is obtained by superposition,

$$x(t) = \int_{-\infty}^{\infty} G(t - \tau)\, f(\tau)\, \mathrm{d}\tau = (G * f)(t).$$

Hence, the delta measure plays a dual role: it represents both a localized initial state in phase space and an idealized instantaneous force in time.

1.2.3 Dirac Delta and the Need for Generalized Functions

Point localization and impulse forcing appear ubiquitously in physics. A point charge at x_0, an impulse force at $t = 0$, or a perfectly sharp initial condition are all modeled by the **Dirac delta** δ. Yet δ is not an ordinary function in any L^p-space: it cannot be evaluated pointwise and has "infinite height and zero width". This tension motivates a framework that extends functions to allow such singular objects while retaining linearity and calculus rules. That framework is the theory of **generalized functions** (distributions).

Definition 1.11: Dirac delta (via action on test functions)

Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a smooth, **compactly supported** test function. The Dirac delta at $x_0 \in \mathbb{R}^n$ is the linear functional

$$\langle \delta_{x_0}, \phi \rangle := \phi(x_0).$$

More generally, a **distribution** on $\mathbb{R}^n$ is a continuous linear functional $T : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$; we write $\langle T, \phi \rangle$ for its action.

Ordinary locally integrable functions $u \in L^1_{\text{loc}}$ embed into this space by

$$\langle T_u, \phi \rangle := \int_{\mathbb{R}^n} u(x)\, \phi(x)\, \mathrm{d}x,$$

so distributions strictly extend functions: δ_{x_0} has no L^1 density, yet is a valid distribution. Equivalently, δ_{x_0} is the **unit point mass** as a Radon measure: for Borel A, $\delta_{x_0}(A) = \mathbf{1}_{\{x_0 \in A\}}$.

Although it is not really a "function", we can still define the concepts of derivative in the sense of distributions by considering how its derivative acts on a function.

Definition 1.12: Distributional derivative

For a distribution T, its derivative $D^\alpha T$ is defined by

$$\langle D^\alpha T, \phi \rangle := (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle, \quad \phi \in C_c^\infty.$$

The reason why we admit this definition is due to the **partial integration**. For the Dirac delta function, we have

$$\langle \delta'_{x_0}, \phi \rangle = -\langle \delta_{x_0}, \phi' \rangle = -\phi'(x_0).$$

If we write it in a more "function-like" way, it will look like a partial integration:

$$\int_{\mathbb{R}^n} \delta'(x - x_0)\phi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} \delta(x - x_0)\phi'(x)\,\mathrm{d}x = -\phi'(x_0),$$

notice that the reason why we require the $\delta\phi$ term vanishes is that ϕ is compact support.

Example 1.5: Heaviside step function

If Θ is the Heaviside step function (sometimes also denoted by H) on $\mathbb{R}$ defined as

$$\langle \Theta, \phi \rangle = \int_0^\infty \phi(x)\,\mathrm{d}x,$$

then $H' = \delta_0$ in the distributional sense:

$$\langle \Theta', \phi \rangle = -\langle \Theta, \phi' \rangle = -\int_0^\infty \phi'(x)\,\mathrm{d}x = \phi(0).$$

Remark 1.2: Basic calculus with δ

For smooth g with $g'(x_0) \neq 0$,

$$\delta\big(g(x)\big) = \sum_{g(x_k)=0} \frac{\delta(x - x_k)}{|g'(x_k)|}, \quad \delta(ax) = \frac{1}{|a|}\delta(x),\ a \neq 0,$$

and δ acts as identity under convolution: $(\delta * \varphi)(x) = \varphi(x)$. These rules are

rigorously justified within distribution theory.

Why distributions are needed (physics).

- **Point localization.** A perfectly known state $x(0) = x_0$ corresponds to the initial measure $\rho_0 = \delta_{x_0}$; under the flow Φ_t, the pushforward is $\rho_t = (\Phi_t)_{\#}\rho_0 = \delta_{\Phi_t(x_0)}$.
- **Impulse forcing.** In a linear time-invariant system $Lx = f$, a unit impulse $f(t) = \delta(t)$ produces the **Green function** G, and general inputs act by convolution $x = G * f$.
- **Jump conditions.** Across interfaces, distributional derivatives encode jumps: if u has a jump of size $[u]_{x_0}$ at x_0, then u' contains $[u]_{x_0}\,\delta_{x_0}$.

Distributions, measures, and densities. Distributions unify smooth densities and singular masses in one linear space. Absolutely continuous measures $\mu(\mathrm{d}x) = \rho(x)\,\mathrm{d}x$ act by $\phi \mapsto \int \phi\rho\,\mathrm{d}x$, while singular measures (e.g., point masses or concentrated sources along submanifolds) are also distributions. This perspective legitimizes "infinitely narrow pulses" as limits in the **weak** topology:

$$\eta_\varepsilon * u \;\to\; u \quad \text{in distributions,}$$

even when u is not an ordinary function.

Example 1.6: Delta transported by dynamics

Let $\dot{x} = f(x)$ with flow Φ_t. Then, for any $\phi \in C_c^\infty$,

$$\big\langle \delta(x - \Phi_t(x_0)), \phi \big\rangle = \phi\big(\Phi_t(x_0)\big),$$

so $\rho_t = \delta(\,\cdot - \Phi_t(x_0))$ solves the continuity equation $\partial_t\rho + \nabla\cdot(f\rho) = 0$ in the distributional sense.

Remark 1.3: Fourier viewpoint

On $\mathbb{R}^n$, the Schwartz space $\mathcal{S}$ and its dual $\mathcal{S}'$ (tempered distributions) accommodate Fourier transforms of distributions:

$$\widehat{\delta_{x_0}}(\xi) = \mathrm{e}^{-\mathrm{i}\xi\cdot x_0} \quad \text{and} \quad \widehat{D^\alpha T}(\xi) = (\mathrm{i}\xi)^\alpha \widehat{T}(\xi),$$

making δ indispensable in spectral and scattering analyses.

In summary, the Dirac delta compels us to enlarge the notion of "function" to distributions: a linear, weak framework in which localization, impulses, and jumps are handled rigorously, and which interfaces seamlessly with measures, Green functions, and operator theory used throughout modern physics.

1.2.4 Metric and Convergence of Measures

The conceptual chain we have established can be summarized as

$$\text{Trajectory} \longrightarrow \text{Density} \longrightarrow \text{Measure } (\delta) \longrightarrow \text{Green Function}.$$

This progression provides the bridge between the geometric view of dynamics and the analytical tools of modern physics.

The Dirac delta is not an ordinary function in any familiar normed space. For example, in the usual L^1 or L^2 norms, any sequence of normalized functions that "concentrate" around a point fails to converge to a function within those spaces. The reason is that in these metrics, the total variation between a narrow pulse and the zero function remains of order one, no matter how narrow the pulse becomes.

To make sense of such limiting processes, we must adopt a weaker notion of convergence. Instead of comparing functions pointwise or by norms, we compare them **through their action on test functions**.

Definition 1.13: Weak convergence

Let $\{\mu_n\}$ be a sequence of measures on $\mathbb{R}^n$, and let μ be another measure. We say that μ_n **converges weakly** to μ if

$$\int_{\mathbb{R}^n} \phi(x)\,\mathrm{d}\mu_n(x) \longrightarrow \int_{\mathbb{R}^n} \phi(x)\,\mathrm{d}\mu(x) \quad \text{for all test functions } \phi \in C_c(\mathbb{R}^n).$$

This notion of convergence can be metrized by, for example, the **Kantorovich–Rubinstein** (or Wasserstein–1) metric:

$$d(\mu_1, \mu_2) = \sup_{|\phi|_{\mathrm{Lip}} \leq 1} \left| \int \phi\,\mathrm{d}\mu_1 - \int \phi\,\mathrm{d}\mu_2 \right|,$$

where the supremum is taken over all Lipschitz test functions with unit Lipschitz constant.

Example 1.7: Approximate identities and weak convergence

Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be an **approximate identity**: $\eta_\varepsilon \geq 0$, $\int \eta_\varepsilon = 1$, and $\operatorname{supp}\eta_\varepsilon \to \{0\}$ as $\varepsilon \to 0$. Examples in one dimension include

$$\eta_\varepsilon(x) = \frac{1}{\sqrt{\pi}\varepsilon} e^{-x^2/\varepsilon^2}, \quad \eta_\varepsilon(x) = \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}, \quad \eta_\varepsilon(x) = \frac{1}{2\varepsilon}\mathbf{1}_{[-\varepsilon,\varepsilon]}(x).$$

Then, for any $\phi \in C_c^\infty$,

$$\int \phi(x)\,\eta_\varepsilon(x - x_0)\,\mathrm{d}x \xrightarrow[\varepsilon\to 0]{} \phi(x_0),$$

i.e. $\eta_\varepsilon(\cdot - x_0)\,\mathrm{d}x \rightharpoonup \delta_{x_0}$ **weakly**, not in L^p-norm.

Within this weaker topology, the delta measure naturally arises as a limit. Let $\delta_\varepsilon(x)$ be any sequence of smooth "approximate identities" satisfying

$$\delta_\varepsilon(x) \geq 0, \quad \int_{\mathbb{R}^n} \delta_\varepsilon(x)\,\mathrm{d}x = 1, \quad \operatorname{supp}(\delta_\varepsilon) \to \{0\} \text{ as } \varepsilon \to 0.$$

Then the associated measures $\mu_\varepsilon(\mathrm{d}x) = \delta_\varepsilon(x)\,\mathrm{d}x$ converge weakly to the Dirac measure δ_0:

$$\int_{\mathbb{R}^n} \phi(x)\,\delta_\varepsilon(x)\,\mathrm{d}x \xrightarrow[\varepsilon\to 0]{} \phi(0).$$

In this sense, the delta function is not a limit in the metric of ordinary function spaces, but a limit in the **space of measures endowed with the weak topology**. This reinterpretation legitimizes the physicist's manipulation of "infinitely narrow" pulses, by viewing them as elements of a larger and better-behaved space.

1.2.5 Measure Theory as the Bridge to Distributions

The modern formulation of integration begins with the notion of a **measure**, a systematic way to assign "size" or "volume" to subsets of $\mathbb{R}^n$.

Definition 1.14: Measure Space

A **measure space** is a triple $(X, \mathcal{A}, \mu)$ where $\mathcal{A}$ is a σ-algebra on X, and $\mu : \mathcal{A} \to [0, \infty]$ satisfies $\mu(\varnothing) = 0$ and countable additivity: if $\{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

Definition 1.15: Lebesgue Measure

For an interval $I \subset \mathbb{R}$, define its **length** by

$$|I| = \begin{cases} b - a, & \text{if } I \subseteq [a, b] \text{ for some } a < b, \\ \infty, & \text{otherwise (i.e., if } I \text{ is unbounded).} \end{cases}$$

To extend this notion to arbitrary sets $A \subset \mathbb{R}$, cover A by a countable family of open intervals $\{I_k\}_{k=1}^{\infty}$ and define the **outer measure**

$$m^*(A) = \inf\left\{ \sum_{k=1}^{\infty} |I_k| \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k,\ I_k \text{ open intervals} \right\}.$$

A set $A \subset \mathbb{R}$ is **Lebesgue measurable** if for every $E \subset \mathbb{R}$,

$$m^*(E) = m^*(E \cap A) + m^*(E \setminus A).$$

Let $\mathcal{L}$ be the collection of all Lebesgue measurable sets. The restriction $m := m^*|_{\mathcal{L}}$ is the **Lebesgue measure**. The triple $(\mathbb{R}, \mathcal{L}, m)$ is called the **Lebesgue measure space** on $\mathbb{R}$; it is complete, translation-invariant, and extends length in the sense that $m((a,b)) = b - a$ for all $a < b$.

Equipped with the Lebesgue measure space $(\mathbb{R}, \mathcal{L}, m)$, we can obtain the **Lebesgue integral** on $\mathbb{R}$ by applying the general definition of the integral with $\mu = m$. In particular, for any measurable $f : \mathbb{R} \to [-\infty, \infty]$,

$$\int_{\mathbb{R}} f \, \mathrm{d}m \quad \text{(often denoted } \int_{\mathbb{R}} f(x) \, \mathrm{d}x\text{)}$$

is defined via simple functions and the supremum construction above; and f is **(Lebesgue) integrable** if and only if $\int_{\mathbb{R}} |f| \, \mathrm{d}m < \infty$.

Definition 1.16: Lebesgue Integral

Let $(X, \mathcal{A}, \mu)$ be a measure space. A function of the form

$$\phi = \sum_{i=1}^{n} a_i \, \mathbf{1}_{E_i},$$

where $n \in \mathbb{N}$, $a_i \geq 0$, and the sets $E_i \in \mathcal{A}$ are pairwise disjoint, is called a (nonnegative) **simple function**. Its integral is defined by

$$\int_X \phi \, \mathrm{d}\mu = \sum_{i=1}^{n} a_i \, \mu(E_i).$$

If $f : X \to [0, \infty]$ is a measurable function, its **Lebesgue integral** is defined as

$$\int_X f \, \mathrm{d}\mu = \sup \left\{ \int_X \phi \, \mathrm{d}\mu \;\middle|\; \phi \text{ a nonnegative simple function with } 0 \leq \phi \leq f \right\}.$$

For a general real-valued measurable function $f : X \to \mathbb{R}$, define

$$f^+(x) = \max\{f(x), 0\}, \qquad f^-(x) = \max\{-f(x), 0\}.$$

Then $|f| = f^+ + f^-$. If

$$\int_X |f|\,\mathrm{d}\mu < \infty, \quad \text{i.e.,} \quad \int_X f^+\,\mathrm{d}\mu < \infty \text{ and } \int_X f^-\,\mathrm{d}\mu < \infty,$$

we say that f is **(Lebesgue) integrable**, and define

$$\int_X f\,\mathrm{d}\mu = \int_X f^+\,\mathrm{d}\mu - \int_X f^-\,\mathrm{d}\mu.$$

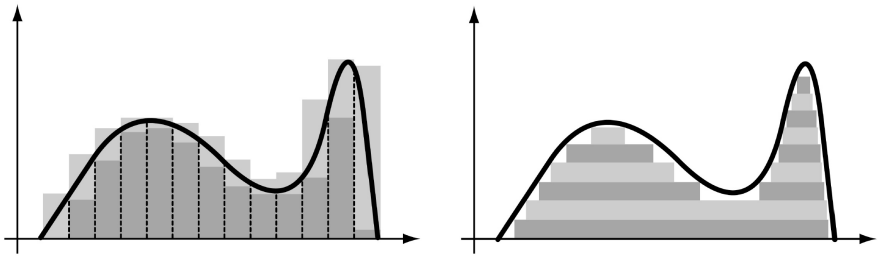

Figure 1.1: The different approximation philosophies for the Riemann (left) and Lebesgue (right) integral.

Unlike the Riemann integral, which partitions the domain, the Lebesgue integral sums over levels of the function's value (Fig. 1.1). It captures oscillatory or discontinuous behavior more naturally.

Example 1.8

The function $f(x) = \sin(1/x)/x$ on $(0, 1]$, with $f(0) = 0$, is not Riemann integrable near $x = 0$, yet it is Lebesgue integrable because the total "mass" $\int_0^1 |f(x)|\,\mathrm{d}x$ is finite. Physically, the signal oscillates infinitely fast but still carries finite energy.

Measures as generalized densities. Ordinary mass densities correspond to absolutely continuous measures of the form $\rho(x)\mathrm{d}x$, but physical systems often

contain discrete or singular components. A unit point mass at x_0 is represented by the **Dirac measure**

$$\delta_{x_0}(A) = \begin{cases} 1, & x_0 \in A, \\ 0, & x_0 \notin A, \end{cases} \qquad \text{so that} \qquad \int_{\mathbb{R}} f(x)\,\mathrm{d}\delta_{x_0}(x) = f(x_0).$$

A periodic array of such point masses,

$$\mu = \sum_{n\in\mathbb{Z}} \delta_n,$$

defines the **Dirac comb**. Its Fourier transform is again a comb, illustrating the duality between spatial periodicity and discrete frequencies.

From measures to distributions. Every finite measure μ defines a linear functional on test functions $\phi \in C_c^\infty$,

$$T_\mu[\phi] = \int \phi(x)\,\mathrm{d}\mu(x),$$

and thus can be regarded as a distribution. The Dirac delta δ is precisely the distribution induced by the Dirac measure. This identification clarifies the connection between measure theory and the calculus of generalized functions.

Summary. Measure theory provides a unified language for both continuous and discrete mass distributions. Lebesgue measure describes the uniform background, Dirac-type measures capture localized structures, and weak convergence connects the discrete and continuous viewpoints. These concepts form the conceptual bridge leading to the theory of distributions, Fourier transforms, and Green functions developed in the following sections.

1.2.6 Concentration and Flow

In dynamical systems, this language of measures provides a unified description of both individual trajectories and statistical ensembles. A single trajectory corresponds to a moving Dirac measure

$$\rho_t = \delta(x - \Phi_t(x_0)),$$

while a distribution of initial conditions corresponds to a smooth density $\rho(x, 0)$. The time evolution of any such measure is expressed by the pushforward

$$\rho_t = (\Phi_t)_{\#}\rho_0, \quad \rho_t(A) = \rho_0\big(\Phi_{-t}(A)\big).$$

Hence, in the language of measures, the dynamics is a flow on the *space of probability measures*. The Dirac measure travels along the deterministic trajectory, while smooth densities are transported and deformed according to the same flow.

This viewpoint will later allow us to interpret the Green function $G(t, t')$ as the kernel of a linear propagator acting on measures, and to connect the response of a system to perturbations with the geometry of its flow.

1.2.7 More Generalized Functions

The Dirac delta is only the simplest member of a large family of **generalized functions** that extend the ordinary notion of a function while preserving linearity and differentiation. Many of these objects arise naturally in physics whenever we idealize sharply localized or discontinuous phenomena. Below, we summarize some of the most common examples.

Example 1.9: Heaviside step function

The **Heaviside function** is defined as

$$\Theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

In distributional sense, its derivative is the Dirac delta:

$$\frac{\mathrm{d}\Theta}{\mathrm{d}x} = \delta(x).$$

Physical meaning: a signal or potential that turns on instantaneously at $x = 0$, such as the activation of a voltage step or a boundary condition switched at a given time.

Example 1.10: Sign function

The **sign function** is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Its derivative is a multiple of the delta function:

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sgn}(x) = 2\,\delta(x).$$

Physical meaning: describes the direction or polarity of a field or force, and appears in the derivative of the absolute value function, $|x|' = \operatorname{sgn}(x)$.

Example 1.11: Principal value distribution

The **Cauchy principal value** p.v.$(1/x)$ is defined by

$$\left\langle \text{p.v.}\tfrac{1}{x}, \phi \right\rangle = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\phi(x)}{x}\, \mathrm{d}x.$$

This distribution captures the symmetric, finite part of the otherwise divergent integral. *Physical meaning:* appears in Hilbert transforms, dispersion relations, and the real part of propagators in quantum and wave theories.

Example 1.12: Derivatives of the delta function

The derivatives $\delta^{(n)}(x)$ are defined by

$$\langle \delta^{(n)}, \phi \rangle = (-1)^n \phi^{(n)}(0).$$

These represent higher-order singular sources. *Physical meaning:* occur in multipole expansions, for instance an electric dipole with moment $\mathbf{p}$ at the origin has

$$\rho(\mathbf{r}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{r}).$$

Example 1.13: Dirac comb (periodic delta array)

The **Dirac comb** is a periodic distribution defined by

$$\text{III}_T(x) = \sum_{n=-\infty}^{\infty} \delta(x - nT).$$

It is self-Fourier up to scaling:

$$\mathcal{F}\{\text{III}_T(x)\} = \frac{2\pi}{T}\, \text{III}_{2\pi/T}(k).$$

Physical meaning: models equally spaced sampling in time or space, such as crystal lattices, diffraction gratings, or pulse trains.

Example 1.14: Surface and line deltas

For a surface $S \subset \mathbb{R}^3$ defined by $g(\mathbf{r}) = 0$ with $\nabla g \neq 0$, the **surface delta** δ_S is given by

$$\int_{\mathbb{R}^3} \phi(\mathbf{r})\, \delta_S(\mathbf{r})\, \mathrm{d}^3 r = \int_S \phi(\mathbf{r})\, \mathrm{d}S = \int_{\mathbb{R}^3} \phi(\mathbf{r})\, \delta(g(\mathbf{r}))\, |\nabla g(\mathbf{r})|\, \mathrm{d}^3\mathbf{r}.$$

Physical meaning: idealized surface or line sources, such as surface charge densities $\rho(\mathbf{r}) = \sigma\, \delta_S(\mathbf{r})$ or current filaments $\mathbf{J}(\mathbf{r}) = I\, \hat{t}\, \delta_L(\mathbf{r})$.

Example 1.15: Approximate identities

A family $\{\eta_\varepsilon\}$ of smooth functions satisfying

$$\eta_\varepsilon \geq 0, \quad \int \eta_\varepsilon = 1, \quad \operatorname{supp}(\eta_\varepsilon) \to \{0\}$$

is called an **approximate identity**. Each η_ε represents a normalized bump of width ε, and in the limit $\eta_\varepsilon \rightharpoonup \delta$ weakly. *Physical meaning:* finite-width pulses or Gaussian beams approaching an ideal point source.

Remark 1.4: Summary

Distribution	**Formal property**	**Physical interpretation**
$\delta(x)$	Localization, $\int \delta = 1$	Point source or impulse
$\Theta(x)$	$H' = \delta$	Sudden switch, step response
$\operatorname{sgn}(x)$	$(\operatorname{sgn})' = 2\delta$	Direction or polarity
p.v.$\frac{1}{x}$	Odd, singular at 0	Symmetric singular field
$\delta^{(n)}$	$(\delta^{(n)})' = \delta^{(n+1)}$	Multipole moment source
$\mathrm{III}_T(x)$	Periodic sum of δ's	Lattice or sampling structure
$\delta_S(\mathbf{r})$	Supported on a surface	Surface (line) charge (current)

These examples illustrate how generalized functions unify discrete, continuous, and singular descriptions within a single analytical framework.

1.3 Fourier Transforms of Generalized Functions

After knowing how and why the Green functions are useful, we now want to ask "how to compute them?" Unfortunately, there is no general method to find them. However, for the cases having translational symmetry the Fourier transform can help us find most of them.

1.3.1 Schwartz Space and Tempered Distributions

The class of **rapidly decreasing smooth functions**, known as the **Schwartz space**, provides the natural setting for Fourier analysis and for defining tempered distributions.

Definition 1.17: Schwartz space

The **Schwartz space** $\mathcal{S}(\mathbb{R}^n)$ consists of all C^∞ functions $f : \mathbb{R}^n \to \mathbb{C}$ such that for all multi-indices α, β,

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty.$$

Equivalently, every derivative of f decays faster than any power of $1/|x|$:

$$|f(x)| = \mathcal{O}(|x|^{-N}) \quad \forall N > 0.$$

The topology of $\mathcal{S}$ is defined by the family of seminorms

$$p_{\alpha,\beta}(f) = \sup_x |x^\alpha \partial^\beta f(x)|.$$

Definition 1.18: Tempered Distributions

The dual space of $\mathcal{S}$, denoted $\mathcal{S}'$, is the space of **tempered distributions**. An element $T \in \mathcal{S}'$ acts on $f \in \mathcal{S}$ by a continuous linear functional:

$$\langle T, f \rangle \in \mathbb{C}.$$

Example 1.16

- Ordinary functions that grow at most polynomially (e.g. $f(x) = x^2$);
- The Dirac delta and its derivatives;
- Principal value distributions such as p.v.$(1/x)$.

Now, we can define the Fourier transform on $\mathcal{S}$ and $\mathcal{S}'$. The Fourier transform preserves $\mathcal{S}$:

$$\mathcal{F} : \mathcal{S} \to \mathcal{S}, \qquad \hat{f}(k) = \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\, dx.$$

Consequently, it extends by duality to $\mathcal{S}'$:

$$\langle \mathcal{F}[T], f \rangle = \langle T, \mathcal{F}[f] \rangle.$$

This allows one to define Fourier transforms of distributions such as

$$\mathcal{F}[\delta](k) = 1, \qquad \mathcal{F}[1](k) = 2\pi\delta(k).$$

Remark 1.5

- $\mathcal{S}$ is dense in $L^2(\mathbb{R}^n)$ and invariant under the Fourier transform.
- Tempered distributions are the natural setting for Green functions in frequency space, since $G(k)$ often behaves like a rational function rather than an L^1 function.

- Many propagators in physics are tempered distributions—e.g. the Feynman propagator is defined through analytic continuation of a tempered distribution.

The Fourier transform extends naturally to distributions. In fact, it often **simplifies** their meaning: singular objects in real space become smooth (or oscillatory) functions in frequency space, and vice versa. This duality between localization and oscillation lies at the heart of wave mechanics, signal theory, and field propagation.

Definition 1.19: Fourier transform of a distribution

For a test function $\phi \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform of a distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ is defined by duality:

$$\langle \widehat{T}, \phi \rangle = \langle T, \widehat{\phi} \rangle, \quad \widehat{\phi}(k) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k\cdot x} \phi(x)\, \mathrm{d}x.$$

Thus every tempered distribution has a well-defined Fourier transform, and the transform is again a tempered distribution.

Example 1.17: Dirac delta

$$\mathcal{F}\{\delta(x - x_0)\} = \mathrm{e}^{-\mathrm{i}kx_0}, \qquad \mathcal{F}\{\delta(x)\} = 1.$$

- **Meaning:** a point source in real space corresponds to a uniform amplitude in frequency space. This expresses the uncertainty principle at the distributional level: perfect localization in x implies complete delocalization in k.

Example 1.18: Constant function

$$\mathcal{F}\{1\} = 2\pi\, \delta(k).$$

- **Meaning:** a uniform field in real space corresponds to zero spatial frequency; all its Fourier weight is concentrated at $k = 0$.

Example 1.19: Heaviside function

In one dimension,

$$\mathcal{F}\{\Theta(x)\} = \pi\delta(k) + \frac{1}{\mathrm{i}k}.$$

The appearance of the singular term $\mathrm{p.v.}(1/k)$ reflects the discontinuity of H.

- **Meaning:** a step in real space decomposes into an infinite superposition of oscillations in k, with amplitudes decaying as $1/k$.

Example 1.20: Principal value distribution

$$\mathcal{F}\left\{\mathrm{p.v.}\frac{1}{x}\right\} = -\mathrm{i}\pi\ \mathrm{sgn}(k).$$

- **Meaning:** the Hilbert transform is a $\pm 90°$ phase shift in Fourier space, embodying causality and analytic continuation (e.g. the Kramers–Kronig relations in optics).

Example 1.21: Derivatives of the delta

For any integer $n \geq 0$,

$$\mathcal{F}\{\delta^{(n)}(x)\} = (\mathrm{i}k)^n.$$

- **Meaning:** differentiation in x corresponds to multiplication by $\mathrm{i}k$ in k, a fundamental property used throughout quantum mechanics and spectral theory.

Example 1.22: Gaussian and approximate identities

The normalized Gaussian

$$\eta_\varepsilon(x) = \frac{1}{\sqrt{2\pi}\varepsilon} \mathrm{e}^{-x^2/(2\varepsilon^2)}$$

has transform

$$\widehat{\eta}_\varepsilon(k) = \mathrm{e}^{-\frac{1}{2}\varepsilon^2 k^2}.$$

As $\varepsilon \to 0$, $\eta_\varepsilon \to \delta$, and correspondingly $\widehat{\eta}_\varepsilon \to 1$.

- **Meaning:** the delta arises as the limit of infinitely broad spectra— the dual manifestation of an infinitely localized pulse.

Example 1.23: Dirac comb

$$\mathcal{F}\{\mathrm{III}_T(x)\} = \frac{2\pi}{T}\,\mathrm{III}_{2\pi/T}(k).$$

- **Meaning:** periodicity in one domain corresponds to discrete sampling in the dual domain. This is the mathematical foundation of diffraction patterns and the Nyquist–Shannon sampling theorem.

Remark 1.6: Summary of key Fourier pairs

Distribution $T(x)$	**Transform** $\widehat{T}(k)$	**Comment**
$\delta(x)$	1	Point $\leftrightarrow$ constant spectrum
1	$2\pi\delta(k)$	Uniform $\leftrightarrow$ zero frequency
$\delta^{(n)}(x)$	$(\mathrm{i}k)^n$	Differentiation $\leftrightarrow$ multiplication
$\Theta(x)$	$\pi\delta(k) + \frac{1}{\mathrm{i}k}$	Step $\leftrightarrow$ $1/k$ decay
p.v.$\frac{1}{x}$	$-\mathrm{i}\pi\ \mathrm{sgn}(k)$	Causality phase shift
$\mathrm{III}_T(x)$	$\frac{2\pi}{T}\,\mathrm{III}_{2\pi/T}(k)$	Periodicity $\leftrightarrow$ discreteness

These examples illustrate that the Fourier transform acts as a bridge between localization and oscillation, connecting the analytic structure of generalized

functions with their physical interpretation as signals, waves, and Green functions.

Remark 1.7: Physical interpretation

In quantum and wave physics, the Fourier duality between x and k is the correspondence between position and momentum, or between temporal and spectral domains. Generalized functions like δ, Θ, and p.v.$(1/x)$ therefore embody idealized limiting cases of physical observables: perfect localization, instantaneous onset, and causal response. This language prepares us for the use of Green functions, where sources and responses are linked by the same transform machinery.

1.3.2 Green Functions as Inverse Fourier Transforms

The Fourier-transform framework provides a powerful way to **construct Green functions explicitly**. For linear differential operators with constant coefficients, the Fourier transform converts differential equations into algebraic ones, so that inversion becomes a simple division in frequency space. The corresponding Green function then arises as the **inverse Fourier transform** of this algebraic inverse.

Definition 1.20: Green function via Fourier transform

Let L be a linear differential operator acting on functions of $x \in \mathbb{R}^n$, and let $G(x)$ be its Green function defined by

$$L\,G(x) = \delta(x).$$

Taking the Fourier transform of both sides gives

$$\widehat{L\,G}(k) = \widehat{\delta}(k) = 1.$$

Because differentiation in x corresponds to multiplication by $\mathrm{i}k$ in k, we obtain

$$\widehat{L\,G}(k) = L(\mathrm{i}k)\,\widehat{G}(k) = 1, \quad \text{so that} \quad \widehat{G}(k) = \frac{1}{L(\mathrm{i}k)}.$$

Hence, the Green function in configuration space is

$$G(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\mathrm{e}^{\mathrm{i}k\cdot x}}{L(\mathrm{i}k)}\,\mathrm{d}^n k,$$

interpreted as an inverse Fourier transform in the sense of distributions.

Example 1.24: Poisson equation in $\mathbb{R}^3$

Consider the Poisson equation for a potential field $\phi(\mathbf{r})$:

$$\nabla^2\phi(\mathbf{r}) = -\rho(\mathbf{r}),$$

whose Green function $G(\mathbf{r})$ satisfies

$$\nabla^2 G(\mathbf{r}) = -\delta(\mathbf{r}).$$

Taking the Fourier transform:

$$-(k^2)\,\widehat{G}(\mathbf{k}) = -1 \implies \widehat{G}(\mathbf{k}) = \frac{1}{k^2}.$$

Therefore,

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3} \int \frac{\mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\mathbf{r}}}{k^2}\,\mathrm{d}^3 k = \frac{1}{4\pi|\mathbf{r}|}.$$

Meaning: the Green function recovers the familiar $1/r$ Coulomb potential, which expresses the influence of a point source in an isotropic medium.

Example 1.25: Wave equation in one dimension

For the operator

$$L = \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2},$$

the Green function $G(x,t)$ satisfies

$$L\,G(x,t) = -\delta(x)\,\delta(t).$$

Fourier transforming in both x and t:

$$(-k^2 + \omega^2/c^2)\,\widehat{G}(k,\omega) = -1, \quad \text{so that} \quad \widehat{G}(k,\omega) = \frac{1}{k^2 - (\omega/c)^2}.$$

The inverse transform gives

$$G(x,t) = \frac{1}{(2\pi)^2}\iint \frac{\mathrm{e}^{\mathrm{i}(kx-\omega t)}}{k^2 - (\omega/c)^2}\,\mathrm{d}k\,\mathrm{d}\omega.$$

The proper choice of contour in ω (enforcing causality through $\omega \to \omega + \mathrm{i}\varepsilon$) selects the **retarded** or **advanced** Green function, corresponding to forward- or backward-in-time propagation.

Remark 1.8: Physical interpretation

In the Fourier domain, the Green function $\widehat{G}(k)$ acts as the **response amplitude** of the system at frequency k, while in real space $G(x)$ represents the **spatial or temporal propagation kernel**. The inverse Fourier transform connects these two views:

$$\text{frequency response} \xleftrightarrow{\mathcal{F}^{-1}} \text{spacetime propagation}.$$

Thus the Green function bridges algebraic inversion and physical causality: the denominator $L(\mathrm{i}k)$ encodes the dispersion relation, and its poles determine the characteristic waves or modes of the system.

Example 1.26: General form of Green function for constant-coefficient operator

For any linear PDE of the form

$$L(\partial_x)\,u(x) = f(x), \quad L(\partial_x) = \sum_\alpha a_\alpha\,\partial_x^\alpha,$$

the solution is obtained by convolution:

$$u(x) = (G * f)(x), \quad G(x) = \frac{1}{(2\pi)^n}\int \frac{\mathrm{e}^{\mathrm{i}k\cdot x}}{L(\mathrm{i}k)}\,\mathrm{d}^n k.$$

In particular:

- For $L = \partial_t + a\,\partial_x$ (advection), $G(x,t) = \delta(x-at)$, representing translation at velocity a.
- For $L = \partial_t^2 - c^2\partial_x^2$ (wave), $G(x,t)$ is supported on the light cone $|x| = c|t|$.
- For $L = \nabla^2 - m^2$ (Yukawa), $G(\mathbf{r}) = -\frac{\mathrm{e}^{-m|\mathbf{r}|}}{4\pi|\mathbf{r}|}$.

Each case corresponds to a distinct physical propagation geometry.

Remark 1.9: Summary

- The Fourier transform converts differential operators to algebraic factors: $\partial_x^n \leftrightarrow (\mathrm{i}k)^n$.
- The Green function is obtained as the inverse Fourier transform of $1/L(ik)$.
- The poles of $1/L(\mathrm{i}k)$ determine the system's resonant or propagating modes.
- Causality is enforced by the $\mathrm{i}\varepsilon$ prescription, which shifts poles off the real axis and selects the retarded solution.

This construction unifies the operator and distribution viewpoints: a Green

function is simply the inverse of a differential operator, represented in either physical or spectral space.

1.3.3 Poles, Dispersion, and Causality

The denominator $L(\mathrm{i}k)$ in the Fourier representation of a Green function encodes the essential physical information of the system: its **modes of oscillation**, **decay rates**, and **causal structure**. The analytic behavior of $1/L(\mathrm{i}k)$ in the complex frequency plane determines how disturbances propagate and how the Green function behaves in time.

Definition 1.21: Poles of the Green function

Let $\widehat{G}(k,\omega) = [L(\mathrm{i}k, -\mathrm{i}\omega)]^{-1}$. A point (k_0, ω_0) where $L(\mathrm{i}k_0, -\mathrm{i}\omega_0) = 0$ is called a **pole** or **characteristic mode** of the system. In the complex ω-plane, the locations of these poles control the temporal behavior of the response.

Example 1.27: Simple harmonic oscillator

Consider the Green function of the damped oscillator:

$$m\ddot{x} + \gamma\dot{x} + kx = \delta(t).$$

Fourier transforming in time:

$$(-m\omega^2 + \mathrm{i}\gamma\omega + k)\,\widehat{G}(\omega) = 1, \quad \widehat{G}(\omega) = \frac{1}{k - m\omega^2 + \mathrm{i}\gamma\omega}.$$

The poles are at

$$\omega_\pm = \frac{\mathrm{i}\gamma}{2m} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}}, \quad \omega_0 = \sqrt{k/m}.$$

Their imaginary parts determine the damping, and their real parts determine the oscillation frequency. The inverse transform yields the familiar damped

response

$$G(t) = \frac{1}{m\omega_d} \mathrm{e}^{-\gamma t/(2m)} \sin(\omega_d t)\, \Theta(t),$$

where ω_d is the damped frequency. *Physical meaning:* each pole represents an eigenmode of the system, and the Green function is their superposition weighted by causal factors.

Example 1.28: Wave equation and dispersion relation

For the wave operator

$$L = \frac{\partial^2}{\partial t^2} - c^2\nabla^2, \quad \widehat{G}(\mathbf{k}, \omega) = \frac{1}{-\omega^2 + c^2 k^2}.$$

The poles occur at $\omega = \pm ck$, which define the **dispersion relation**. Each Fourier component propagates with phase velocity $v_p = \omega/k = c$, and the real-space Green function is supported on the light cone:

$$|\mathbf{r}| = c|t|.$$

Physical meaning: the singularities in $\widehat{G}$ specify the characteristic waves permitted by the system.

Example 1.29: Causality and the $i\epsilon$ prescription

In practice, the physical Green function must respect causality: no effect can precede its cause. Mathematically, we enforce this by displacing the poles slightly off the real axis:

$$\widehat{G}_{\mathrm{ret}}(\omega) = \frac{1}{L(-\mathrm{i}\omega + \mathrm{i}\varepsilon)}, \quad \varepsilon > 0.$$

This ensures that the inverse Fourier transform vanishes for $t < 0$, producing the **retarded Green function**. The opposite sign gives the **advanced Green function**. *Physical meaning:* the infinitesimal imaginary shift encodes time

ordering and guarantees that signals propagate only forward in time.

Example 1.30: Resonance and response amplitude

If a forcing term oscillates at frequency ω_f, the steady-state response amplitude is

$$|\widehat{G}(\omega_f)| = \frac{1}{|L(-\mathrm{i}\omega_f)|}.$$

When ω_f approaches the real part of a pole, the amplitude grows sharply—this is **resonance**. In damped systems the pole's imaginary part prevents divergence, yielding a finite peak width proportional to the decay rate. *Physical meaning:* resonant peaks in $\widehat{G}(\omega)$ correspond to long-lived modes or quasi-stationary states in the time domain.

Geometric viewpoint. Each pole of $\widehat{G}$ corresponds to a solution of the **characteristic equation** $L(\mathrm{i}k) = 0$, defining the dispersion surface in (k, ω)-space. The geometry of this surface determines the allowed propagation directions and group velocities. For relativistic wave equations, it coincides with the light cone $\omega^2 = c^2k^2$; for dispersive media, it deforms into more general forms, reflecting how phase velocity varies with frequency.

From the analytic structure of $\widehat{G}(k, \omega)$ we can read off every essential physical property of the system: causality, damping, resonance, and wave propagation. Thus, in the Fourier domain,

$$\textbf{Poles} \longleftrightarrow \textbf{Modes},$$

$$\textbf{Imaginary shifts} \longleftrightarrow \textbf{Causality},$$

$$\textbf{Residues} \longleftrightarrow \textbf{Energy or amplitude.}$$

This spectral viewpoint completes the bridge between the operator formulation of Green functions and their physical interpretation as propagators of signals and fields.

1.3.4 The Residue Theorem and Contour Integration

The analytic structure of $\widehat{G}(\omega)$ in the complex ω-plane not only encodes causality but also provides a practical method to **evaluate** the Green function in the time domain. By contour integration and the residue theorem, we can compute inverse Fourier transforms by summing over the contributions of the poles.

Definition 1.22: Inverse Fourier transform in time

For a one-dimensional system (or after spatial Fourier transform), the Green function in time is given by

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{G}(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega.$$

If $\widehat{G}(\omega)$ extends analytically to the complex ω-plane with isolated poles, this integral can be evaluated by closing the contour and applying the residue theorem.

Example 1.31: Damped oscillator revisited

For the oscillator Green function

$$\widehat{G}(\omega) = \frac{1}{m(\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega/m)}.$$

The poles are at $\omega_\pm = \pm\omega_d - \mathrm{i}\gamma/(2m)$. To obtain the **retarded Green function**, valid for $t > 0$, we close the contour in the lower half-plane (where $\operatorname{Im}\omega < 0$ so that $\mathrm{e}^{-\mathrm{i}\omega t}$ decays). By the residue theorem,

$$G(t) = 2\pi\mathrm{i} \sum_{\text{poles in lower half-plane}} \operatorname{Res}\left[\widehat{G}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}, \omega_p\right].$$

Carrying out the residues yields

$$G(t) = \frac{1}{m\omega_d}\, \mathrm{e}^{-\gamma t/(2m)} \sin(\omega_d t)\, \Theta(t),$$

which agrees with the time-domain solution derived earlier.

Example 1.32: Causal Green function for the wave operator

For the 1D wave equation,

$$\widehat{G}(k,\omega) = \frac{1}{(\omega + \mathrm{i}0)^2 - c^2k^2}.$$

Closing the contour in the lower half-plane for $t > 0$, the poles at $\omega = \pm ck - i0$ yield

$$G(k,t) = \frac{\sin(ckt)}{ck}\, \Theta(t).$$

Taking the inverse Fourier transform in k gives

$$G(x,t) = \frac{1}{2c}\, \Theta(t)\, [\delta(x - ct) + \delta(x + ct)],$$

which describes right- and left-moving signals at speed c. *Physical meaning:* the contour integral automatically enforces the light-cone propagation dictated by causality.

Remark 1.10: Contour choice and causality

- For $t > 0$ (retarded case), close the contour in the lower half-plane.
- For $t < 0$ (advanced case), close it in the upper half-plane.
- The small imaginary shift $\omega \to \omega + \mathrm{i}\epsilon$ selects which set of poles contributes.

Geometrically, the contour choice corresponds to **time-ordering** of cause and effect. In quantum field theory, this is the origin of the *Feynman propagator*:

$$G_F(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega^2 - \omega_0^2 + \mathrm{i}\varepsilon}\,\mathrm{d}\omega,$$

where the poles are symmetrically displaced to ensure both causal directions are included under time ordering.

Example 1.33: Feynman propagator in frequency space

The contour prescription can be visualized as

$$\frac{1}{\omega^2 - \omega_0^2 + \mathrm{i}\varepsilon} = \mathcal{P}\left(\frac{1}{\omega^2 - \omega_0^2}\right) - \mathrm{i}\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)],$$

which separates the propagator into a **principal value** (dispersive) and a **delta** (absorptive) part. This decomposition reflects the division between **real** and **imaginary** response, or equivalently between **energy storage** and **energy dissipation**.

Example 1.34: Wick rotation

In many applications—particularly in quantum field theory and statistical mechanics—one deforms the time contour to imaginary time via the **Wick rotation**

$$t \to -\mathrm{i}\tau, \quad \omega \to \mathrm{i}\omega_E.$$

Then

$$G_E(\tau) = \frac{1}{2\pi}\int \frac{\mathrm{e}^{\mathrm{i}\omega_E \tau}}{\omega_E^2 + \omega_0^2}\,\mathrm{d}\omega_E,$$

which is exponentially decaying rather than oscillatory.

- **Physical meaning:** this transforms the Lorentzian propagator into an *Euclidean Green function*, suitable for equilibrium or statistical for-

mulations, and forms the basis of path-integral methods in imaginary time.

Remark 1.11: Summary of the residue method

1. Identify the poles of $\widehat{G}(\omega)$ in the complex plane.
2. Choose a contour consistent with the desired causality ($t > 0$ or $t < 0$).
3. Evaluate the residues at enclosed poles.
4. Sum the contributions and multiply by $2\pi\mathrm{i}$.

This simple geometric rule in the complex plane translates directly into physically meaningful propagation in spacetime.

Physical interpretation. Each pole represents a mode that contributes an exponential or oscillatory term $\mathrm{e}^{-\mathrm{i}\omega_p t}$ in time. The contour integral *selects* which modes can propagate according to causality. In dissipative systems, poles below the real axis correspond to decaying signals; in unstable ones, poles above correspond to growth. The residue theorem thus provides both a computational and a conceptual bridge between complex analysis and physical propagation.

Contour integration turns the inversion of operators into geometry: causal propagation follows from which poles lie inside a chosen half-plane. The small shift $\mathrm{i}\varepsilon$ determines direction of time, and Wick rotation connects dynamical propagation with statistical equilibrium. Through the residue theorem, the abstract analytic structure of $\widehat{G}(\omega)$ becomes a vivid picture of waves, damping, and causality in the complex plane.

Chapter 2

Green Operators in Mechanics and Field Theory

The concepts developed in the preceding chapters— delta measures, Green functions, and variational principles— find direct and powerful applications in mechanics and field theory. In this chapter we show how Green operators describe the propagation of disturbances in continuous media, the oscillations of discrete mechanical systems, and the response of relativistic fields.

2.1 Green Functions as Propagators on the Space of Measures

The Green function is a central concept in both analysis and physics. It represents the fundamental response of a linear system to a localized source. Within the measure-theoretic view of dynamics, a Green function can be interpreted as the *propagator* that transports an infinitesimal perturbation or measure from one point in time to another.

2.1.1 Green Functions in Dynamical Systems

We introduce the concept of the Green function from the **linearized dynamics**.

Consider a general (possibly nonlinear) dynamical system

$$\dot{\mathbf{x}} = f(x), \in \mathbb{R}^n,$$

and let $\Phi_t(x_0)$ denote its flow map, so that $x(t) = \Phi_t(x_0)$. A small perturbation $\xi(t)$ around a reference trajectory $x_0(t)$ satisfies the linearized (variational) equation

$$\dot{\xi}(t) = Df(x_0(t))\,\xi(t) + \eta(t),$$

where Df is the Jacobian matrix of f, and $\eta(t)$ represents a small external forcing.

The associated **Green function** $G(t, t')$ is the matrix-valued kernel that satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t, t') = Df(x_0(t))\,G(t, t'), \quad G(t', t') = I,$$

where I is the identity matrix. Formally, this Green function is the derivative of the flow map:

$$G(t, t') = D\Phi_{t,t'}(x_0) = \frac{\partial \Phi_t(x_0)}{\partial x_0}.$$

For any forcing term $\eta(t)$, the perturbation can be expressed as

$$\xi(t) = \int_{t_0}^{t} G(t, t')\,\eta(t')\,\mathrm{d}t'.$$

In particular, a delta impulse at $t = t_1$,

$$\eta(t) = \delta(t - t_1)\,\eta_0,$$

produces the response

$$\xi(t) = G(t, t_1)\,\eta_0.$$

Thus, $G(t, t')$ propagates an instantaneous disturbance forward in time.

2.1.2 Green Function as a Propagator on Measures

From the perspective of measures, consider a family of probability measures ρ_t that evolve according to the flow:

$$\rho_t = (\Phi_t)_{\#}\rho_0.$$

If the initial measure is a single Dirac mass $\rho_0 = \delta(x - x_0)$, then

$$\rho_t = \delta(x - \Phi_t(x_0)).$$

A small variation of the initial point $x_0 \mapsto x_0 + \epsilon\xi_0$ induces a variation of the measure given to the first order by

$$\delta\rho_t = -\nabla \cdot \big(G(t,0)\,\xi_0\,\delta(x - \Phi_t(x_0))\big).$$

Hence, the Green function not only propagates perturbations of state vectors, but also describes how infinitesimal deformations of measures evolve under the deterministic flow.

2.1.3 Connection to Physical Response

In the context of linear physical systems, the Green function $G(t,t')$ plays the role of the **impulse response function**. It relates an external force $f(t)$ to the system's displacement:

$$x(t) = \int_{-\infty}^{t} G(t,t')\,f(t')\,\mathrm{d}t'.$$

From a dynamical viewpoint, this convolution integral is the continuous analogue of the pushforward of measures: each infinitesimal "impulse" of input at time t' is transported forward by the propagator $G(t,t')$.

Therefore, the measure-theoretic and the dynamical-system perspectives merge naturally:

the Dirac delta represents localization in space or time, while the Green function represents the deterministic flow of that localization.

This interpretation provides the conceptual foundation for later chapters, where we will use Green functions to analyze both temporal evolution (via convolution) and spatial propagation (in the study of differential operators and fields).

2.2 Green Functions in Linear Systems

2.2.1 The Discrete Oscillator Chain

Consider a one-dimensional chain of N identical masses m connected by springs of stiffness k. Let $x_i(t)$ denote the displacement of the i-th mass from equilibrium. The equations of motion are

$$m\ddot{x}_i + k(2x_i - x_{i+1} - x_{i-1}) = f_i(t), \quad i = 1, \dots, N,$$

with appropriate boundary conditions (e.g., fixed ends). In matrix form this reads

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{f}(t),$$

where $M = mI$ and K is the tridiagonal stiffness matrix.

The Green function $G_{ij}(t - t')$ satisfies

$$M\ddot{G}_{ij}(t - t') + KG_{ij}(t - t') = \delta_{ij}\,\delta(t - t'),$$

and the solution for any driving force is

$$x_i(t) = \sum_j \int G_{ij}(t - t')\, f_j(t')\, \mathrm{d}t'.$$

Diagonalizing the stiffness matrix, $K = S\Lambda S^{-1}$, with eigenvalues $\lambda_\alpha = m\omega_\alpha^2$ and normal modes $S_{\alpha i}$, we obtain

$$G_{ij}(t) = \sum_{\alpha} \frac{S_{\alpha i} S_{\alpha j}}{m\omega_\alpha} \sin(\omega_\alpha t)\, \Theta(t),$$

a discrete analog of the continuous Green function for the wave equation. This explicitly shows that G_{ij} is the propagator of mechanical influence through the coupled system.

2.2.2 Continuum Limit and the Wave Equation

Letting the lattice spacing $a \to 0$ and defining $x_i(t) \to u(x,t)$, the difference operator becomes the spatial derivative, and the discrete equation tends to

$$\rho \frac{\partial^2 u}{\partial t^2} - T \frac{\partial^2 u}{\partial x^2} = f(x,t),$$

where ρ is the mass density and T the tension. This is the one-dimensional wave equation. The corresponding Green function satisfies

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) G(x,t;x',t') = -\frac{1}{T}\,\delta(x-x')\,\delta(t-t'), \quad c = \sqrt{T/\rho}.$$

In analogy with the spacetime formulation, the retarded solution is

$$G(x,t;x',t') = \frac{1}{2\rho c}\Theta\left(t - t' - \frac{|x-x'|}{c} \right),$$

representing the causal propagation of mechanical disturbances along the string.

2.3 Green Functions in Spacetime and Wave Propagation

The Green function formalism reaches its full power when applied to wave phenomena, where both space and time are involved. In this context, the Green function

encodes how a localized disturbance at a spacetime point $(\mathbf{r}', t')$ propagates through space and time according to a wave equation. The interplay of causality, propagation speed, and boundary conditions determines its precise form.

2.3.1 The Wave Equation

Consider the scalar wave equation in three-dimensional space:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\mathbf{r}, t) = -s(\mathbf{r}, t),$$

where $u(\mathbf{r}, t)$ represents a field (such as the acoustic or electromagnetic potential) and $s(\mathbf{r}, t)$ represents a source distribution. The associated Green function $G(\mathbf{r}, t; \mathbf{r}', t')$ is defined by

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t; \mathbf{r}', t') = -\delta(\mathbf{r} - \mathbf{r}')\,\delta(t - t').$$

Once G is known, the field can be obtained by superposition:

$$u(\mathbf{r}, t) = \iint G(\mathbf{r}, t; \mathbf{r}', t')\, s(\mathbf{r}', t')\, \mathrm{d}^3\mathbf{r}'\, \mathrm{d}t'.$$

2.3.2 Causality and the Retarded Green Function

Because waves cannot propagate instantaneously, the physically meaningful solution must satisfy causality: a disturbance at time t' can only influence later times $t > t'$. This condition selects the **retarded** Green function, which vanishes for $t < t'$.

In free space, the retarded solution is

$$G_{\mathrm{ret}}(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta\left(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r} - \mathbf{r}'|}\,\Theta(t - t'),$$

where Θ is the Heaviside step function. This expression states that a signal emitted from $\mathbf{r}'$ at time t' arrives at $\mathbf{r}$ precisely after the travel time $|\mathbf{r} - \mathbf{r}'|/c$.

Similarly, the **advanced** Green function, nonzero only for $t < t'$, is given by

$$G_{\text{adv}}(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta\left(t' - t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r} - \mathbf{r}'|}\,\Theta(t' - t).$$

In physical applications, we usually select the retarded Green function, corresponding to forward-in-time propagation.

2.3.3 Frequency-Domain Representation

Taking the temporal Fourier transform,

$$\hat{G}(\mathbf{r}, \mathbf{r}'; \omega) = \int_{-\infty}^{\infty} G(\mathbf{r}, t; \mathbf{r}', t')\, \mathrm{e}^{\mathrm{i}\omega(t-t')}\, \mathrm{d}(t - t')\,,$$

we obtain the Helmholtz equation

$$\left(\nabla^2 + k^2\right) \hat{G}(\mathbf{r}, \mathbf{r}'; \omega) = -\delta(\mathbf{r} - \mathbf{r}'), \quad k = \frac{\omega}{c}.$$

The outgoing-wave (retarded) solution in free space is

$$\hat{G}(\mathbf{r}, \mathbf{r}'; \omega) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}.$$

Inverse transforming with respect to ω recovers the retarded time-domain function above. Thus, the frequency-domain Green function represents the steady-state spatial response at frequency ω, while the time-domain version encodes causal propagation.

Remark 2.1: Geometric interpretation

The delta function in time localizes the source in the temporal coordinate, and the Green function spreads this localization along the **light cone** of the spacetime metric:

$$|\mathbf{r} - \mathbf{r}'| = c(t - t').$$

Hence, the Green function is concentrated on the surface of constant propagation time between source and observer. In relativistic notation, the d'Alembertian operator

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

acts on Minkowski spacetime, and the equation

$$\Box G(x, x') = \delta^{(4)}(x - x')$$

defines the fundamental solution whose support lies on the light cone. The retarded and advanced solutions correspond to causal and anti-causal propagation along that cone.

Remark 2.2: Physical meaning

The spacetime Green function expresses a universal physical principle: **local causes produce effects that propagate continuously and causally through the geometry of space and time.** In this sense,

- the delta function represents a localized event in spacetime;
- the Green function represents the propagation of that event's influence;
- the operator L (or $\Box$) encodes the geometry and medium through which propagation occurs.

This completes the transition from discrete trajectories and temporal impulses to continuous fields and spacetime propagation. The following chapters will extend these ideas to Green functions in general linear operators, Green's identities, and the variational formulations that connect them to the principles of mechanics and field theory.

Example 2.1: Volume Change and Invariant Densities

Consider $\dot{x} = Ax$ with flow $\Phi_t = \mathrm{e}^{tA}$. Then $\det D\Phi_t = \mathrm{e}^{t\,\mathrm{Tr}A}$ and

$$\rho_t(y) = \rho_0\big(\mathrm{e}^{-tA}y\big)\,\mathrm{e}^{-t\,\mathrm{Tr}A}.$$

Hence Lebesgue measure is invariant iff $\mathrm{Tr}A = 0$. For Hamiltonian flows, the symplectic structure implies $\mathrm{Tr}A = 0$ along trajectories, so Liouville measure is invariant.

2.4 Boundary Value Problems

In many physical systems, differential equations are accompanied not only by source terms but also by **boundary conditions** that specify the behavior of the field at the edge of the domain. The role of the Green function is to encode both the differential operator and the boundary conditions, thus fully determining the solution.

2.4.1 General Formulation

Consider a linear differential operator L acting on $u(\mathbf{r})$ over a domain Ω, with boundary $\partial\Omega$:

$$L\,u(\mathbf{r}) = f(\mathbf{r}), \quad \mathbf{r} \in \Omega.$$

The boundary conditions are expressed through a linear operator B:

$$B\,u(\mathbf{r}) = g(\mathbf{r}), \quad \mathbf{r} \in \partial\Omega.$$

There are three usual boundary conditions:

- **Dirichlet condition:** $u|_{\partial\Omega} = g_D$. The value of the function is prescribed on the boundary.

- **Neumann condition:** $\partial_n u|_{\partial\Omega} = g_N$. The normal derivative is specified instead.
- **Mixed (Robin) condition:** $\alpha u + \beta\, \partial_n u = g_R$. Linear combination of both.

The Green function $G(\mathbf{r}, \mathbf{r}')$ satisfies

$$L\, G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

with boundary condition

$$B\, G(\mathbf{r}, \mathbf{r}') = 0, \quad \mathbf{r} \in \partial\Omega.$$

The solution of the boundary value problem can then be written as

$$u(\mathbf{r}) = \int_\Omega G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}')\, \mathrm{d}^3\mathbf{r}' + \int_{\partial\Omega} \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial u}{\partial n'} - u(\mathbf{r}') \frac{\partial G}{\partial n'} \right] \mathrm{d}\mathbf{S}'.$$

The second integral ensures that the boundary condition is properly enforced.

Example 2.2: Poisson Equation in a Half-Space

Let us solve

$$\nabla^2 \phi = -\frac{\rho(\mathbf{r})}{\varepsilon_0}, \quad z > 0,$$

with Dirichlet boundary condition $\phi(x, y, 0) = 0$.
We can construct the Green function by the *method of images*:

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} - \frac{1}{4\pi|\mathbf{r} - \mathbf{r}''|}, \quad \mathbf{r}'' = (x', y', -z').$$

The second term represents the potential of an image charge located below the plane, ensuring $G = 0$ on $z = 0$.
Then

$$\phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \int_{z'>0} G(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}')\, \mathrm{d}^3\mathbf{r}'.$$

Remark 2.3

- The boundary term in the integral representation determines whether we are enforcing Dirichlet or Neumann conditions.
- For time-dependent equations (e.g. wave or diffusion), similar constructions apply but require causal or retarded Green functions.
- The method of images is valid for simple geometries where a mirror symmetry exists; for more general shapes, one must construct G numerically or by eigenfunction expansion.

2.4.2 Method of Images from the Green Functions

The method of images can be interpreted as the construction of a Green function that satisfies a prescribed boundary condition. Instead of introducing image charges by intuition, we can see them as mathematical terms ensuring that the Green function vanishes (or satisfies the desired derivative condition) on the boundary.

Green function for a half-space conductor

For a grounded conducting plane at $z = 0$, the boundary condition is

$$G(\mathbf{r}, \mathbf{r}') = 0 \quad \text{for } z = 0.$$

We look for G satisfying

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}').$$

In free space, the fundamental solution is

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}.$$

To impose the boundary condition, we can modify G_0 by adding a suitable homogeneous solution of Laplace's equation:

$$G(\mathbf{r},\mathbf{r}') = G_0(\mathbf{r},\mathbf{r}') + G_h(\mathbf{r},\mathbf{r}'), \quad \nabla^2 G_h = 0.$$

We require $G(x,y,0;\mathbf{r}') = 0$. A simple choice is

$$G_h(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}''|}, \quad \mathbf{r}'' = (x',y',-z').$$

Then

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} - \frac{1}{4\pi|\mathbf{r}-\mathbf{r}''|}.$$

This is precisely the Green function for the Dirichlet boundary condition in the half-space.

Remark 2.4: Interpretation as an image source

The second term corresponds to an image source located at the mirror point $\mathbf{r}''$ with opposite sign. When used in Poisson's equation,

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}, \quad \phi|_{z=0} = 0,$$

the solution becomes

$$\phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\int_{z'>0} G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'.$$

The image term ensures that the potential vanishes on the boundary for any charge distribution in $z > 0$. Thus, the image charge is nothing more than a manifestation of the boundary-adapted Green function.

Remark 2.5: Other boundary conditions

Analogously, for a Neumann boundary condition $\partial_n G = 0$ on $z = 0$, the appropriate Green function is obtained by *adding* instead of subtracting the

image term:

$$G_N(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi|\mathbf{r}-\mathbf{r}''|}.$$

This corresponds to an image of the **same sign**.

From the Green function perspective, the method of images is a constructive way to find a Green function that satisfies the correct boundary conditions:

$$\text{Find } G = G_0 + G_h \text{ such that } LG = \delta, \quad BG = 0 \text{ on } \partial\Omega.$$

Whenever the geometry is simple enough to guess an analytic G_h, the image method provides a fast and physically transparent solution.

The heat equation in a half-space

The image construction also appears naturally in time-dependent diffusion or heat-conduction problems. Consider the one-dimensional heat equation on the half-line $x > 0$:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0, \qquad u(0,t) = 0, \qquad u(x,0) = f(x).$$

The free-space Green function (heat kernel) satisfies

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right) G_0(x,t;x',t') = \delta(x-x')\,\delta(t-t'),$$

and is given by

$$G_0(x,t;x',t') = \frac{\Theta(t-t')}{\sqrt{4\pi\kappa(t-t')}} \exp\left[-\frac{(x-x')^2}{4\kappa(t-t')}\right].$$

To enforce the Dirichlet boundary condition $u(0,t) = 0$, we add a homogeneous (boundary-correcting) term G_h so that $G = G_0 + G_h$ satisfies $G(0,t;x',t') = 0$. Choosing

$$G_h(x,t;x',t') = -G_0(x,t;-x',t'),$$

we obtain

$$G(x,t;x',t') = \frac{\Theta(t-t')}{\sqrt{4\pi\kappa(t-t')}}\left[\exp\left(-\frac{(x-x')^2}{4\kappa(t-t')}\right) - \exp\left(-\frac{(x+x')^2}{4\kappa(t-t')}\right)\right].$$

This Green function is equivalent to introducing an **image heat source** of opposite sign located at the mirror point $-x'$. It automatically enforces $u(0,t) = 0$ for all $t > t'$.

The solution is therefore

$$u(x,t) = \int_0^\infty G(x,t;x',0)\, f(x')\, \mathrm{d}x'.$$

Physically, the image term represents the "heat deficit" needed to cancel any flux crossing the insulated or grounded boundary.

Remark 2.6

- The structure of G is parallel to the electrostatic image potential: both are obtained by adding a mirror source with appropriate sign.
- For Neumann boundary conditions $\partial_x u|_{x=0} = 0$, the Green function is obtained by **adding** the image term instead of subtracting it.
- This technique generalizes to other linear PDEs—diffusion, sound, elasticity—whenever the domain possesses reflection symmetry.

For Neumann boundary conditions, the normal derivative of the field is specified on the boundary, rather than its value:

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g_N.$$

From the Green function perspective, we now require

$$\left.\frac{\partial G}{\partial n}\right|_{\partial\Omega} = 0.$$

In the half-space geometry with boundary $z = 0$, the corresponding Green function can again be built by adding a homogeneous solution:

$$G_N(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi|\mathbf{r} - \mathbf{r}''|}, \quad \mathbf{r}'' = (x', y', -z').$$

This "same-sign" image ensures that the normal derivative vanishes at the plane:

$$\left.\frac{\partial G_N}{\partial z}\right|_{z=0} = 0.$$

Neumann boundary conditions. The Neumann condition corresponds physically to an insulated boundary (no flux) in diffusion, or to a perfectly reflecting surface for waves. For an arbitrary source $f(\mathbf{r}')$, the solution reads

$$u(\mathbf{r}) = \int_\Omega G_N(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') \, \mathrm{d}^3 r',$$

which guarantees $\partial_n u = 0$ on $z = 0$.

In the Dirichlet case, we **subtract** the image, enforcing $u = 0$. In the Neumann case we **add** the image, enforcing $\partial_n u = 0$. These two constructions are the simplest illustrations of how Green functions encode different boundary operators.

Spherical boundary. Consider now Laplace's equation outside a grounded conducting sphere of radius a:

$$\nabla^2 \phi = 0, \quad \phi|_{r=a} = 0,$$

with a point charge q located at $\mathbf{r}_0$ ($|\mathbf{r}_0| > a$). We seek a Green function that vanishes on the spherical boundary.

The Green function satisfying the Dirichlet condition can be constructed as

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|} - \frac{a/r_0}{4\pi|\mathbf{r} - (a^2/r_0^2)\mathbf{r}_0|}.$$

The second term represents an **image charge**

$$q' = -\,q\,\frac{a}{r_0}, \quad \mathbf{r}_0' = \frac{a^2}{r_0^2}\mathbf{r}_0,$$

located inside the sphere. This choice guarantees $G(a,\theta) = 0$ for all angles.

In terms of the Green function formalism, we can verify

$$\nabla^2 G = -\delta(\mathbf{r} - \mathbf{r}_0), \quad G|_{r=a} = 0.$$

Thus the potential in the region $r > a$ due to an arbitrary charge density ρ is

$$\phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\int_{r'>a} G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}',$$

and the induced surface charge on the sphere follows from $-\varepsilon_0 \partial_r \phi|_{r=a}$.

Remark 2.7

- The spherical image method corresponds mathematically to constructing the Green function with respect to a *Kelvin transformation* $r \mapsto a^2/r$.
- For Neumann conditions on the sphere, the image charge changes sign and magnitude, adjusted to make $\partial_r G = 0$ at $r = a$.
- Both planar and spherical cases illustrate the general rule:

$$G = G_0 + G_h, \quad \begin{cases} G_h = -G_0(\text{reflected}) & \text{Dirichlet,} \\ G_h = +G_0(\text{reflected}) & \text{Neumann.} \end{cases}$$

2.5 Field Theoretic Interpretation

Many field equations of physics derive from an action principle

$$S[\phi] = \int \mathcal{L}(\phi, \partial_\mu \phi)\,\mathrm{d}^4 x,$$

where $\mathcal{L}$ is the Lagrangian density. The Euler–Lagrange equation for the field $\phi(x)$ is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = J(x),$$

where $J(x)$ is an external source. Linearizing around a background configuration yields a linear operator L acting on ϕ:

$$L\phi(x) = J(x).$$

The Green function satisfies

$$LG(x, x') = \delta^{(4)}(x - x'),$$

and the field solution follows as

$$\phi(x) = \int G(x, x')\, J(x')\, \mathrm{d}^4 x'.$$

Hence, the Green function is the kernel of the inverse operator L^{-1} that maps sources to fields.

Example 2.3: The Klein–Gordon field

For the scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\, \partial^\mu \phi - m^2 \phi^2\right) + J\phi,$$

the equation of motion is

$$(\Box + m^2)\phi(x) = J(x),$$

where $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$. The Green function satisfies

$$(\Box + m^2)G(x, x') = \delta^{(4)}(x - x').$$

Fourier transforming,

$$\tilde{G}(k) = \frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon},$$

where the infinitesimal term $i\varepsilon$ specifies the causal (retarded or Feynman) prescription. The corresponding spacetime Green function represents the propagation amplitude of a scalar excitation of mass m from point x' to x. In quantum field theory, this same object becomes the two-point correlation function $\langle 0|T\{\phi(x)\phi(x')\}|0\rangle$.

The Green operator thus plays a universal role:

- In **mechanics**, it describes the temporal response of oscillators and continuous media to impulses and forces.
- In **field theory**, it represents the propagation kernel relating sources and fields across spacetime.
- In **variational mechanics**, it is the kernel of the operator obtained by extremizing the action.

From the delta measure that localizes an event to the Green function that propagates its influence, the same mathematical structure underlies both classical and relativistic dynamics.

Chapter 3

Operator Methods and the Functional Viewpoint

Up to this point we have treated the Green function as an integral kernel relating sources and responses. We now adopt a more abstract but unifying perspective: the Green function is the integral kernel of an inverse operator acting on a function space. This operator approach provides the foundation for quantum mechanics, spectral theory, and functional analysis.

3.1 Linear Operators and Their Inverses

Let L be a linear operator acting on a Hilbert space $\mathcal{H}$ of square-integrable functions,

$$L : \mathcal{H} \to \mathcal{H}, \quad Lf = g.$$

If L is invertible, its inverse L^{-1} satisfies

$$L^{-1}L = LL^{-1} = I.$$

When L is a differential operator, L^{-1} can be represented by an integral kernel $G(x, x')$:

$$(L^{-1}f)(x) = \int G(x, x')\, f(x')\, \mathrm{d}x'.$$

Applying L to both sides gives

$$LG(x, x') = \delta(x - x'),$$

which is precisely the defining relation of the Green function. Thus, the Green function is the **coordinate representation of the inverse operator** L^{-1}.

Remark 3.1

Sometimes, the Green functions may be written as the "inverse of the operator"

$$L^{-1} = G(x, x').$$

Strictly speaking, this is incorrect since Green functions are the "kernels" of the inverses, but it is still quite common.

Distributional Kernels and the Schwartz Space. Many operator kernels are not ordinary functions but **distributions**, i.e., the Green functions can themselves be distributions but not functions. For example,

$$\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \quad \text{and} \quad \frac{\delta\left(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r} - \mathbf{r}'|}\,\Theta(t - t'),$$

which are the Green functions for the Laplacian and the d'Alenbert operator, respectively.

More generally, operators acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ have kernels in the dual space $\mathcal{S}'$, the tempered distributions. This framework provides a rigorous justification for Fourier transforms, contour integrals, and δ-function manipulations used throughout physics.

Bounded and Unbounded Operators. Most physically relevant operators, especially differential ones, are *unbounded*: their norm $\|Lf\|$ can grow arbitrarily large even if $\|f\| = 1$. Hence, specifying the domain $\mathcal{D}(L) \subset \mathcal{H}$ is essential.

Example 3.1: Momentum and Hamiltonian operators

The momentum operator $\hat{p} = -\mathrm{i}\hbar\,\partial_x$ is symmetric on smooth compactly supported functions, but becomes truly self-adjoint only after fixing boundary conditions (e.g. vanishing at infinity or periodic). Self-adjointness guarantees the spectral theorem and ensures that $\mathrm{e}^{-\mathrm{i}Lt}$ is unitary, corresponding physically to probability conservation.

Self-adjointness is therefore not merely a mathematical nicety but a statement of physical consistency.

Example 3.2: Poisson's equation and the Laplace operator

In electrostatics, the potential $\phi(\mathbf{r})$ due to a charge density $\rho(\mathbf{r})$ satisfies

$$L\phi = -\nabla^2\phi = \frac{\rho}{\varepsilon_0}.$$

The operator $L = -\nabla^2$ is the Laplacian on $\mathbb{R}^3$. Formally,

$$\phi(\mathbf{r}) = L^{-1}\rho = \int G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\,\mathrm{d}^3r',$$

where

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

is the Green function of the Laplace operator. Thus, the Coulomb potential is the kernel of the inverse Laplacian.

Example 3.3: Wave operator and the retarded Green function

For wave propagation in free space,

$$L = \Box = \frac{1}{c^2}\partial_t^2 - \nabla^2, \quad LG = \delta(\mathbf{r} - \mathbf{r}')\,\delta(t - t').$$

The causal (retarded) Green function is

$$G_{\text{ret}}(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta(t - t' - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|}\,\Theta(t - t'),$$

which describes a signal emitted at $(\mathbf{r}', t')$ that reaches $(\mathbf{r}, t)$ after the light-travel time $|\mathbf{r} - \mathbf{r}'|/c$.

3.2 Spectral Viewpoint

When studying a linear operator L, it is often not enough to know how to solve $LG = \delta$ formally. We want to understand *how* the operator acts, which parts of the system dominate its response, and how the underlying modes contribute to physical observables. The **spectral viewpoint** provides precisely this insight: it decomposes the operator into its elementary excitations (eigenmodes) and expresses the Green function as their weighted superposition. This perspective links algebraic structure to physical behavior, revealing stability, resonance, and long-time dynamics in a unified framework.

From the completeness of eigenfunctions $\{u_n\}_{n=1}^{\infty}$, one can write

$$\delta(x - x') = \sum_n u_n(x) u_n^*(x'),$$

so the Green function naturally expands as

$$G(x, x') = \sum_n \frac{u_n(x) u_n^*(x')}{\lambda_n}.$$

This expression already shows that each eigenmode contributes inversely to its eigenvalue—low-lying modes dominate the response.

Resolvent Green Functions. While the discrete spectral sum is illuminating, realistic operators may have both discrete and continuous spectra, or depend on external parameters (like energy). In such cases, it is more powerful to study a one-parameter family of inverses:

$$R(\lambda) = (L - \lambda I)^{-1},$$

called the **resolvent operator**, which is the **analytic continuation** of the Green functions.

Just like in analytic number theory, we use the Riemann ζ function to analyze the discrete prime numbers analytically. Introducing $R(\lambda)$ extends the analysis into the complex λ-plane, allowing us to treat both discrete and continuous structures in an analytic framework. Its poles mark eigenvalues, and its singular structure captures the full spectrum of L. Thus, rather than summing over modes explicitly, we can recover all spectral information from the analytic properties of the resolvent.

The corresponding kernel

$$G_\lambda(x, x') = \langle x|R(\lambda)|x'\rangle$$

is the **resolvent Green function**, which satisfies

$$G_\lambda(x, x') = \sum_n \frac{u_n(x)\, u_n^*(x')}{\lambda_n - \lambda}.$$

As $\lambda \to 0$, the resolvent reduces to the ordinary Green function:

$$G(x, x') = G_{\lambda=0}(x, x') = L^{-1}(x, x').$$

Hence, the resolvent formalism not only generalizes the notion of inverse, but also connects spectral analysis with complex analysis.

Spectral theorem formulation. For a self-adjoint L, the spectral theorem guarantees the existence of a projection-valued measure $E(\lambda)$ on $\mathbb{R}$ such that

$$L = \int_{\sigma(L)} \lambda \, \mathrm{d}E(\lambda), \quad f(L) = \int_{\sigma(L)} f(\lambda) \, \mathrm{d}E(\lambda),$$

and thus

$$R(\lambda) = (L - \lambda I)^{-1} = \int_{\sigma(L)} \frac{1}{\mu - \lambda} \, \mathrm{d}E(\mu).$$

This integral representation emphasizes that discrete spectra yield isolated poles, while continuous spectra generate branch cuts—physically corresponding to bound and scattering states, respectively.

When the spectrum is continuous, the discrete expansion becomes an integral over continuum states:

$$G_\lambda(x, x') = \int \frac{u_k(x) u_k^*(x')}{E(k) - \lambda} \, \mathrm{d}k.$$

The infinitesimal term $+\mathrm{i}\varepsilon$ specifies the radiation (outgoing) condition, ensuring causal and physically meaningful solutions.

3.2.1 Connections to Quantum Mechanics

In quantum mechanics, the resolvent

$$R(E) = (E - H + \mathrm{i}0^+)^{-1}, \qquad G_E(\mathbf{r}, \mathbf{r}') = \langle \mathbf{r} | R(E) | \mathbf{r}' \rangle$$

packages **all** spectral information of the Hamiltonian H into an analytic object on the complex energy plane. Poles on the real axis identify bound states; branch cuts encode the scattering continuum; isolated poles in the lower half-plane correspond to resonances (quasi-bound states with finite lifetimes).

From spectrum to propagation By completeness of eigenstates $\{\psi_n\}$ (plus continuum $\{\psi_{E,\alpha}\}$),

$$G_E(\mathbf{r},\mathbf{r}') = \sum_n \frac{\psi_n(\mathbf{r})\psi_n^*(\mathbf{r}')}{E_n - E - \mathrm{i}0^+} + \int \mathrm{d}E' \sum_\alpha \frac{\psi_{E',\alpha}(\mathbf{r})\psi_{E',\alpha}^*(\mathbf{r}')}{E' - E - \mathrm{i}0^+}.$$

Low-lying modes dominate the response; the $+\mathrm{i}0^+$ selects the retarded (causal) solution.

Time evolution is recovered by inverse transform:

$$U(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht} = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \mathrm{d}E\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Et}\, R(E),$$

so residues of $R(E)$ govern long-time behavior (bound states $\Rightarrow$ persistent oscillations; resonances $\Rightarrow$ exponential decay).

- **Density of states.**

  $$\rho(E) = -\frac{1}{\pi}\,\mathrm{Im}\,\mathrm{Tr}\; G^R(E), \quad \rho(\mathbf{r},E) = -\frac{1}{\pi}\,\mathrm{Im}\; G_E^R(\mathbf{r},\mathbf{r}).$$

  Thus, imaging $\mathrm{Im}\; G^R$ maps out available quantum states (STM/STS, impurities, edges).

- **Spectral function and sum rules.**

  $$A(\mathbf{r},\mathbf{r}';E) \equiv -2\;\mathrm{Im}\; G_E^R(\mathbf{r},\mathbf{r}'), \quad \int \frac{\mathrm{d}E}{2\pi}\, A = \delta(\mathbf{r}-\mathbf{r}').$$

  A is the energy-resolved probability amplitude that underlies photoemission and transport.

- **Scattering (Lippmann–Schwinger and T-matrix).** With $H = H_0 + V$,

  $$|\psi^{(+)}\rangle = |\phi\rangle + G_0^{(+)}(E)\, V\, |\psi^{(+)}\rangle, \qquad T(E) = V + V\, G_0^{(+)}(E)\, T(E).$$

  Here $G_0^{(+)}(E)$ is the free resolvent; all scattering amplitudes and cross sections are built from $T(E)$, hence from $G_0^{(+)}$.

- **Linear response (Kubo).** Dissipative response is the imaginary part of a retarded correlator, i.e. a resolvent of a commutator:

$$\chi^R_{AB}(\omega) = \frac{1}{\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\langle[A(t), B(0)]\rangle = \langle\langle A; B\rangle\rangle^R_\omega,$$

  and real or imaginary parts are tied by Kramers–Kronig due to analyticity.

- **Interacting systems and self-energy.** Dyson equation in energy space:

$$G^R(E) = \left[E - H_0 - \Sigma^R(E)\right]^{-1},$$

  so interactions are compressed into the self-energy Σ (level shifts and lifetimes), again via a resolvent.

How the resolvent enforces causality? The prescription $E \to E + \mathrm{i}0^+$ selects the retarded Green function G^R, guaranteeing outgoing (radiation) boundary conditions and causal propagation. Advanced solutions use $-\mathrm{i}0^+$. Different choices correspond to physically distinct experiments (response and preparation).

3.3 Semigroup and Functional Calculus Viewpoint

More generally, functions of operators can be defined through spectral theory. If L has eigenpairs (λ_n, u_n), then

$$f(L) = \sum_n f(\lambda_n)\, |u_n\rangle\langle u_n|.$$

The exponential operator

$$\mathrm{e}^{-\mathrm{i}Lt} = \sum_n \mathrm{e}^{-\mathrm{i}\lambda_n t}\, |u_n\rangle\langle u_n|$$

acts as a time-evolution operator. Its kernel

$$K(x,t;x',0) = \langle x|\mathrm{e}^{-\mathrm{i}Lt}|x'\rangle$$

is the **propagator** or time-dependent Green function. This formulation unifies dynamical evolution, spectral expansion, and the inverse-operator view into a single framework.

For positive self-adjoint operators $L \geq 0$, the exponential e^{-tL} defines a strongly continuous semigroup satisfying

$$\mathrm{e}^{-(t+s)L} = \mathrm{e}^{-tL}\mathrm{e}^{-sL}, \quad \mathrm{e}^{0L} = I.$$

Its kernel $K_t(x,x') = \langle x|e^{-tL}|x'\rangle$, the **heat kernel**, solves

$$(\partial_t + L)K_t(x,x') = 0, \quad K_{t=0}(x,x') = \delta(x-x').$$

The Green function is its Laplace transform:

$$L^{-1} = \int_0^\infty \mathrm{e}^{-tL}\,\mathrm{d}t, \quad G(x,x') = \int_0^\infty K_t(x,x')\,\mathrm{d}t.$$

Analytic continuation $t \to \mathrm{i}t/\hbar$ turns the heat kernel into the Feynman propagator, connecting Euclidean diffusion with quantum evolution. This relation underlies the path-integral representation of quantum mechanics.

Summary.

- $G(x,x')$ is the kernel of the inverse operator L^{-1}.
- The resolvent $R(\lambda)$ generalizes L^{-1} and encodes the spectrum of L.
- The spectral theorem provides the measure-theoretic foundation linking $R(\lambda)$ and eigen-decomposition.
- The time-evolution operator $e^{-\mathrm{i}Lt}$ yields the propagator $K(x,t;x',0)$.

- The heat kernel e^{-tL} and its Laplace transform connect diffusion, Green functions, and quantum amplitudes.

In this functional language, the Green function becomes the geometric manifestation of operator inversion, while its analytic and spectral structure encodes causality, stability, and dynamics. It forms the unifying thread linking differential equations, spectral theory, and quantum evolution.

Chapter 4

Functional Integrals and Path Representations

The operator formalism developed in the previous chapter reveals that dynamical evolution, Green functions, and spectral decompositions can all be viewed as operations on function spaces. We now take one more conceptual step: to represent these operator exponentials not as abstract algebraic objects, but as *integrals over trajectories* in configuration space. This passage from operators to path integrals is one of the deepest unifications in theoretical physics.

4.1 From Operators to Path Integrals

Consider the time-evolution operator for a quantum particle with Hamiltonian $H = \frac{p^2}{2m} + V(x)$:

$$U(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}.$$

Its kernel

$$K(x_b, t_b; x_a, t_a) = \langle x_b | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t_b - t_a)} | x_a \rangle$$

gives the transition amplitude for a particle initially at x_a to be found at x_b after time $t_b - t_a$.

Using the Trotter product formula,

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(T+V)t} = \lim_{N\to\infty}\left(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V\,t/N}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}T\,t/N}\right)^{N},$$

we can insert complete sets of position states between each factor and obtain an N-fold integral over intermediate positions:

$$K(x_b,t_b;x_a,t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N-1}\mathrm{d}x_j\prod_{j=0}^{N-1}\langle x_{j+1}|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\Delta t}|x_j\rangle.$$

Evaluating each short-time kernel to leading order gives

$$\langle x_{j+1}|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\Delta t}|x_j\rangle \approx \left(\frac{m}{2\pi\mathrm{i}\hbar\Delta t}\right)^{1/2}\exp\left[\frac{\mathrm{i}}{\hbar}\Delta t\left(\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\Delta t}\right)^{2}-V(x_j)\right)\right].$$

Multiplying these factors and taking the continuum limit, we arrive at the celebrated **Feynman path integral**:

$$K(x_b,t_b;x_a,t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b}\exp\left[\frac{\mathrm{i}}{\hbar}S[x(t)]\right]\mathcal{D}x(t),$$

where the action functional is

$$S[x(t)] = \int_{t_a}^{t_b}\left[\tfrac{1}{2}m\dot{x}^2 - V(x)\right]\mathrm{d}t.$$

The Green function of the Schrödinger operator is thus represented as an integral over all possible trajectories weighted by the phase factor $\mathrm{e}^{\mathrm{i}S/\hbar}$.

If we perform a Wick rotation $t \mapsto -\mathrm{i}\tau$, the oscillatory factor $\mathrm{e}^{\mathrm{i}S/\hbar}$ becomes a real exponential $\mathrm{e}^{-S_E/\hbar}$, where S_E is the Euclidean action:

$$S_E[x(\tau)] = \int\left[\tfrac{1}{2}m\dot{x}^2 + V(x)\right]\mathrm{d}\tau.$$

The propagator becomes

$$K_E(x_b, \tau_b; x_a, \tau_a) = \int \exp\left[-\frac{1}{\hbar} S_E[x(\tau)]\right] \mathcal{D}x(\tau),$$

which coincides with the heat kernel

$$K_E = \mathrm{e}^{-(\tau_b - \tau_a)H/\hbar}.$$

Thus, the path integral provides a probabilistic interpretation of the operator semigroup e^{-tL} introduced in the previous chapter: quantum evolution in imaginary time is equivalent to diffusion governed by the same generator.

4.2 Field Theory and Functional Integration

In the limit $\hbar \to 0$, the rapidly oscillating phase $\mathrm{e}^{\mathrm{i}S/\hbar}$ causes destructive interference except near stationary points of $S[x]$, where $\delta S = 0$. These stationary paths are precisely the classical trajectories, and the leading approximation to the kernel is

$$K(x_b, t_b; x_a, t_a) \approx \sum_{\text{classical paths}} A(x_a, x_b)\, \exp\left[\frac{\mathrm{i}}{\hbar} S_{\mathrm{cl}}(x_a, x_b)\right],$$

where A is the semiclassical prefactor. This gives the correspondence principle: the path integral reproduces classical mechanics as the stationary-phase limit of quantum evolution.

The same reasoning extends from particle paths to fields. For a scalar field $\phi(\mathbf{r}, t)$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\nabla \phi)^2 - V(\phi),$$

the generating functional is

$$Z[J] = \int \exp\left[\frac{\mathrm{i}}{\hbar} \int (\mathcal{L} + J\phi)\, \mathrm{d}^4 x\right] \mathcal{D}\phi,$$

and the Green functions of the field are obtained as functional derivatives:

$$G_n(x_1,\dots,x_n) = \frac{1}{\mathrm{i}^n}\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}.$$

Thus, the operator-based Green function formalism extends seamlessly to the infinite-dimensional setting of fields, where it becomes the backbone of quantum field theory and statistical mechanics.

4.2.1 Correlation Functions and Generating Functionals

In the path-integral framework, expectation values of observables become functional averages. For a quantum system with Euclidean action $S_E[\phi]$, we define the partition function

$$Z = \int \mathrm{e}^{-S_E[\phi]/\hbar}\,\mathcal{D}\phi,$$

and the expectation value of any functional $F[\phi]$ as

$$\langle F[\phi]\rangle = \frac{1}{Z}\int F[\phi]\,\mathrm{e}^{-S_E[\phi]/\hbar}\,\mathcal{D}\phi.$$

To generate correlation functions systematically, we introduce an external source $J(x)$ coupled linearly to the field:

$$Z[J] = \int \exp\!\left[-\frac{1}{\hbar}\big(S_E[\phi] - \int J\phi\,\mathrm{d}^d x\big)\right]\mathcal{D}\phi.$$

Then, the n-point correlation functions follow by functional differentiation:

$$G_n(x_1,\dots,x_n) = \frac{1}{Z[0]}\,\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}.$$

In the language of operators, these G_n correspond to time-ordered expectation values $\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle$. Thus, the generating functional $Z[J]$ plays a role analogous to the resolvent or Green operator of the previous chapter, but now in the space of field configurations.

Example 4.1: Free theories

For a free scalar field with quadratic action

$$S_E[\phi] = \frac{1}{2}\int \phi(x)\, L\, \phi(x)\, \mathrm{d}^d x, \quad L = (-\nabla^2 + m^2),$$

the functional integral is Gaussian and can be evaluated exactly:

$$Z[J] = Z[0]\, \exp\left[\frac{1}{2\hbar}\int J(x)\, G(x,x')\, J(x')\, \mathrm{d}^d x\, \mathrm{d}^d x'\right],$$

where G satisfies

$$L\, G(x,x') = \delta(x-x').$$

This $G(x,x')$ is precisely the Green function introduced before: it represents the two-point correlation function,

$$\langle \phi(x)\phi(x')\rangle = \hbar\, G(x,x').$$

Hence, in the free theory, the entire hierarchy of correlations is encoded in the operator inverse L^{-1}, making explicit the continuity between operator and functional viewpoints.

4.2.2 Perturbation Theory and Feynman Diagrams

When the action contains nonlinear terms, e.g.

$$S_E[\phi] = S_{\text{free}}[\phi] + \int V(\phi)\, \mathrm{d}^d x,$$

we expand the interaction exponential as a power series:

$$Z[J] = \int \mathrm{e}^{-S_{\text{free}}[\phi]/\hbar}\left(1 - \frac{1}{\hbar}\int V(\phi) + \frac{1}{2!\hbar^2}\int V(\phi)V(\phi') + \cdots\right)\mathrm{e}^{\frac{1}{\hbar}\int J\phi}\mathcal{D}\phi.$$

Since the free part is Gaussian, all higher-order correlations can be computed using **Wick's theorem**: pairwise contractions of fields are replaced by propagators

$G(x, x')$. Each term in the expansion corresponds to a graph— a **Feynman diagram**—whose edges represent propagators and whose vertices correspond to powers of $V(\phi)$. Thus, perturbation theory in field theory is nothing but the systematic expansion of the generating functional in powers of the interaction, with combinatorics governed by the Gaussian measure.

4.2.3 Connected and Effective Functionals

It is often convenient to define the **connected generating functional**

$$W[J] = \hbar \ln Z[J],$$

whose derivatives generate connected correlation functions:

$$\frac{\delta^2 W[J]}{\delta J(x)\, \delta J(x')} = \langle \phi(x)\phi(x') \rangle_c.$$

The **effective action** $\Gamma[\phi_c]$ is obtained as the Legendre transform of $W[J]$:

$$\Gamma[\phi_c] = W[J] - \int J(x)\phi_c(x)\, \mathrm{d}^d x, \quad \phi_c(x) = \frac{\delta W[J]}{\delta J(x)}.$$

Stationary points of Γ yield the expectation values of fields,

$$\frac{\delta \Gamma[\phi_c]}{\delta \phi_c(x)} = 0,$$

analogous to classical equations of motion but now incorporating quantum corrections. This structure parallels the operator resolvent $R(\lambda)$ as the generator of nonlinear response.

4.3 Statistical Mechanics Analogy

In Euclidean time, the generating functional becomes identical to a partition function:

$$Z = \int \mathrm{e}^{-\beta H[\phi]}\, \mathcal{D}\phi, \quad \beta = 1/k_B T.$$

Here the Green function represents a correlation function in a thermal ensemble, and the operator L acts as the generator of diffusion or relaxation. The correspondence

$$t \leftrightarrow -\mathrm{i}\tau, \qquad \hbar \leftrightarrow k_B T$$

links quantum dynamics to classical statistical mechanics: quantum fluctuations and thermal noise share the same mathematical origin in the path-integral measure.

When interactions are present, ultraviolet divergences arise from short-distance behavior of the propagator $G(x, x')$. These infinities reflect the necessity of redefining the parameters of the theory (mass, coupling, etc.) as functions of a cutoff scale. In the functional language, renormalization corresponds to a flow of the effective action $\Gamma[\phi_c]$ under coarse-graining transformations:

$$\Lambda \frac{\partial \Gamma_\Lambda}{\partial \Lambda} = \mathcal{R}[\Gamma_\Lambda],$$

where $\mathcal{R}$ encodes how integrating out high-frequency modes modifies the dynamics at long wavelengths. This renormalization-group (RG) viewpoint extends the operator idea of spectral decomposition to an infinite hierarchy of scales.

Summary. The functional-integral framework unifies many ideas previously developed:

- Green functions are correlation functions of fluctuating fields.
- Operator inverses become Gaussian integrals over configurations.
- The generating functional $Z[J]$ generalizes the resolvent; its logarithm $W[J]$ encodes connected responses, and its Legendre transform $\Gamma[\phi_c]$ plays the role of an effective dynamical generator.
- Perturbation theory appears as a diagrammatic expansion around the Gaussian measure defined by L^{-1}.

- In Euclidean time, path integrals become statistical partition functions, linking quantum and thermal fluctuations.

From this point on, the Green function ceases to be merely a solution to a differential equation. It becomes the carrier of correlations, spectra, and causality— the unifying mathematical object of both quantum dynamics and statistical field theory.

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