Group Meeting #1

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Outline

1. Methodology
   - From Vector to Tensor
   - From Tensor to Network
   - Tensor Network Approach
2. Application
   - Application in Stochastic Dynamics
   - Example: The SIS Model
   - My idea: Random Walk

From Vector to Tensor

Definition

Vector Space

*Let F be a field. A F-***vector space** *is set V equipped with two operators*

$$+ : V \times V \to V,$$

$$\cdot : F \times V \to V$$

satisfies the following conditions

- $(V, +)$ *is an abelian group,*
- $a \cdot (b \cdot v) = (ab) \cdot v$ *for all* $a,\ b \in F$ *and* $v \in V$,
- $(a + b) \cdot v = a \cdot v + b \cdot v$ *for all* $a,\ b \in F$ *and* $v \in V$,
- $a(u + v) = au + av$ *for all* $a \in F$ *and* $u,\ v \in V$,
- $id_F \in F$ *such that* $id_F \cdot v = v$.

From Vector to Tensor

Definition

Tensor product

Let V and W be vector spaces and I be a subspace of $\langle V \times W \rangle$ spanned by all the vectors having the form

$$(u+v,w)-(u,w)-(v,w),\ (cv,w)-c(v,w),$$

$$(u,v+w)-(u,w)-(u,v),\ (v,cw)-c(v,w).$$

The **tensor product of vector spaces** *V and W denoted by $V \otimes W$ is defined as*

$$\langle V \times W \rangle \,/\, \mathrm{span}(I).$$

From Vector to Tensor

Definition

Tensor product

Let $v \in V$ and $w \in W$. The **tensor product of vectors** *v and w denoted by $v \otimes w$ is*

$$v \otimes w = (v, w).$$

Let $A,\ B \subseteq V \otimes W$. The **tensor product of subsets** *A and B denoted by $A \otimes B$ is*

$$A \otimes B = \{\alpha \otimes \beta | \alpha \in A,\ \beta \in B\}.$$

Remark

Let α and β be bases for V and W respectively. It is easy to check that $\alpha \otimes \beta$ is the basis for $V \otimes W$. We immediately have,
$\dim V \otimes W = \dim V \times \dim W$. *This is why we use the word "tensor".*

From Vector to Tensor

Definition

Let V, V_2, W_1 and W_2 be vector spaces and $f : V_1 \to V_2$ and $g : W_1 \to W_2$ be linear transformations. The **tensor product of linear transformations** *f and g is defined as*

$$f \otimes g : V_1 \otimes W_1 \to V_2 \otimes W_2, \; v \otimes w \mapsto f(v) \otimes g(w).$$

From Tensor to Network

1. The linear transformations and vectors are equivalent to matrices and column vectors in a finite-dimensional vector space.

$$f_i{}^j v_j = w_i \text{ (matrices form)} \leftrightarrow f(v) = w \text{ (abstract form)}.$$

2. The linear transformations and therefore matrices also form vector spaces. We can therefore consider the tensor product of these vector spaces.
3. The tensor product of linear transformations and vectors can be represented by "high-dimensional" matrices (tensor) $M_{i_1 i_2 \cdots i_N}$ in the tensor product of finite-dimensional vector spaces.

From Tensor to Network

Example

Let V_i, W_i, and U be vector spaces, $f : V_1 \to V_2$ and $g : W_1 \to W_2$ be linear transformations, and $u \in U$ be a vector. The tensor product of them is

$$u \otimes f \otimes g \equiv T.$$

T is a linear transformation defined as

$$T : V_1 \otimes W_1 \to U \otimes V_2 \otimes W_2, \; v \otimes w \mapsto u \otimes v \otimes w.$$

This has the tensor form

$$T(v \otimes w) = u \otimes f(v) \otimes g(w)$$

$$\updownarrow$$

$$T_{ijk\alpha\beta}[v \otimes w]^{\alpha\beta} = [u \otimes f(v) \otimes g(w)]_{ijk}.$$

From Tensor to Network

Example

Note that $f : V_1 \to V_2$, $g : W_1 \to W_2$ *and* $u \in U$. *We have the index sets of the tensor* $T_{ijk\alpha\beta}$

$$\alpha \in \{1, \cdots, \dim V_1\},\ \beta \in \{1, \cdots, \dim W_1\},$$

$$i \in \{1, \cdots, \dim U\},\ j \in \{1, \cdots, \dim V_2\},\ k \in \{1, \cdots, \dim W_2\}.$$

From Tensor to Network

Definition

Tensor Diagram Notation

A rank-n tensor T can be represented by a node with n edges corresponding to n indices. The summation of two tensors over some indices can be represented by connecting the corresponding edges.

From Tensor to Network

Example

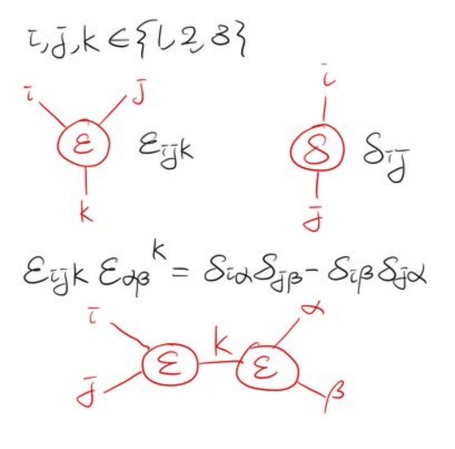


Figure: Levi-Civita symbol.

Tensor Network Approach

1. According to quantum postulate. The state space of a composite quantum system is the tensor product of the state spaces of its individual components.
2. Consider a one-dimensional quantum chain, and the Hilbert space for the ith lattice is $\mathcal{H}_i$, then the corresponding Hilbert space is

$$\bigotimes_{i=1}^{N} \mathcal{H}_i .$$

3. The states in this space can be written as

$$|\Psi\rangle = \sum_{\{s_i\}} \Psi_{s_1 \cdots s_N} |s_1\rangle \otimes \cdots \otimes |s_N\rangle .$$

Tensor Network Approach

1. Note that $\Psi_{s_1 \cdots s_N}$ is a tensor, by the SVD, we can rewrite it as a product of tensors

$$\Psi_{s_1 \cdots s_N} = T^{[1]}{}_{s_1 i_1} T^{[2]}{}_{s_1}{}^{i_1}{}_{i_2} \cdots T^{[N-1]}{}_{s_{N-1}}{}^{i_{N-2}}{}_{i_{N-1}} T^{[N]}{}_{s_N}{}^{i_{N-1}}.$$

   This is called the *Matrix Product State*.

2. The complexity of $\Psi_{s_1 \cdots s_N}$ is

$$\mathcal{O}(\dim \mathcal{H}_i^N)$$

   and that for $T^{[1]}{}_{s_1 i_1} \cdots T^{[N]}{}_{s_N}{}^{i_{N-1}}$ is

$$\mathcal{O}(\dim \mathcal{H}_i).$$

Tensor Network Approach

Example

Greenberger-Horne-Zeilinger (GHZ) state

The GHZ state is

$$
\begin{aligned}
|GHZ\rangle &= \frac{1}{\sqrt{2}}|0\rangle \otimes \cdots \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes \cdots \otimes |1\rangle \\
&= \frac{1}{\sqrt{2}} \sum_{\{s_i\}} \Psi_{s_1 \cdots s_N} \bigotimes_{i=1}^{N} |s_i\rangle ,
\end{aligned}
$$

where

$$
\Psi_{s_1 \cdots s_N} = \begin{cases} 1, & \textit{if } s_1 = \cdots = s_N \in \{0, 1\} \\ 0, & \textit{otherwise.} \end{cases}
$$

Tensor Network Approach

Example

Greenberger-Horne-Zeilinger (GHZ) state (continued)

The MPS form is given by the tensor

$$\frac{1}{\sqrt{2}} Tr \begin{pmatrix} \hat{e}_1 & 0 \\ 0 & \hat{e}_2 \end{pmatrix}^N = \frac{1}{\sqrt{2}} (\hat{e}_1{}^{\otimes N} + \hat{e}_2{}^{\otimes N}).$$

That is,

$$\begin{aligned} |GHZ\rangle &= \frac{1}{\sqrt{2}} \sum_{\{s_i\}} \Psi_{s_1 \cdots s_N} \bigotimes_{i=1}^{N} |s_i\rangle \\ &= \frac{1}{\sqrt{2}} \sum_{\{s_i\}} (\hat{e}_1{}^{\otimes N} + \hat{e}_2{}^{\otimes N})_{s_1 \cdots s_N} \bigotimes_{i=1}^{N} |s_i\rangle . \end{aligned}$$

Application in Stochastic Dynamics

1. **Step 1:** Write the configuration of the state space as a vector $|\Psi(t)\rangle$ and therefore becomes a quantum-mechanics form. Expand this vector with basis $|s_1, s_2, \cdots, s_N\rangle$. That is, describe the state by a vector
$$|\Psi(t)\rangle = \sum_{\{s_i\}} \Psi_{s_1,s_2,\cdots,s_N}(t)\,|s_1, s_2, \cdots, s_N\rangle\,.$$
2. **Step 2:** Write down the "Hamiltonian" H for the system, in other words, find the operator determining the master equation
$$\partial_t |\Psi\rangle = H|\Psi\rangle$$
of the system.

Application in Stochastic Dynamics

1. **Step 3:** Represent the state $|\Psi\rangle$ as MPS. That is,

$$|\Psi\rangle = \sum_{\{s_i\}} \prod_i T^{s_i} |s_1, \cdots, s_N\rangle .$$

2. **Step 4:** Employ the TN algorithm, for example, DMRG or VMPS, to solve the corresponding question. See the example.

Example

Example

SIS Model

1. **Step 1:** *The configuration of the state space as a vector* $|P(t)\rangle$ *and therefore becomes a quantum-mechanics form. Expand this vector with basis* $|s_1, s_2, \cdots, s_N\rangle$*, where* $s_i \in \{0, 1\}$*. That is, describe the state by a vector*

$$|P(t)\rangle = \sum_{\{s_i\}} P_{s_1,s_2,\cdots,s_N}(t) \, |s_1, s_2, \cdots, s_N\rangle \, .$$

2. **Step 2:** *The operator determining the master equation is the infinitesimal Markov Generator* $\hat{W}$

$$\partial_t \, |P\rangle = \hat{W} \, |P\rangle \, .$$

of the system.

Example

Example

SIS Model (continued)

1. **Step 3:** *Represent the state $|P\rangle$ as MPS. That is,*

$$|P\rangle = \sum_{\{s_i\}} \prod_i T^{s_i} |s_1, \cdots, s_N\rangle .$$

2. **Step 4:** *Employing the variational approach, one can see the high-accuracy critical behavior.*

Random Walk

1. For unrestricted random walk. The ensemble average position of the end of the polymer $\vec{r}$ satisfies

$$\langle r \rangle = 0 \text{ and } \langle r^2 \rangle^{1/2} \propto N^{1/2},$$

where N is the number of links in the chain.

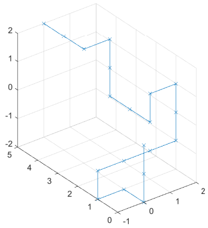
2. The variance of r diverges as $N \to \infty$ and

$$\sqrt{\langle r^2 \rangle - \langle r \rangle^2} \propto N^{1/2}.$$

3. It is therefore natural to view the unrestricted random walk as a critical system with the critical exponent $\nu = 1/2$.

Random Walk

1. Self-Avoid Random Walk is a crucial model in polymer physics.
2. A point move on the lattice $\mathbb{Z}^d$ randomly but without intersection.
3. It can be simulated by the Monte Carlo method.



4. The critical exponent for SARW is $\nu = 1/(2+d)$ if $d = 2, 3$.

Random Walk

1. RW is also a lattice model having critical behaviors.
2. The computational complexity for the simulation of SARW grows very fast as the number of steps increases.
3. Can we also apply the TN approach to it to find the critical behaviors as what we get in the SIS model?