Group Meeting #2

Hao-Yang Yen

NTHU, Interdisciplinary Program of Sciences

2024/7/18

Outline

1. The SIS Model
   - Introduction to the SIS model.
   - Non-equilibrium phase transition.
2. Theoretical Basis of the TN approach
   - From Ising model to the SIS model.
   - The tensor product formulation.
   - The large deviation principle.

Introduction to the SIS Model

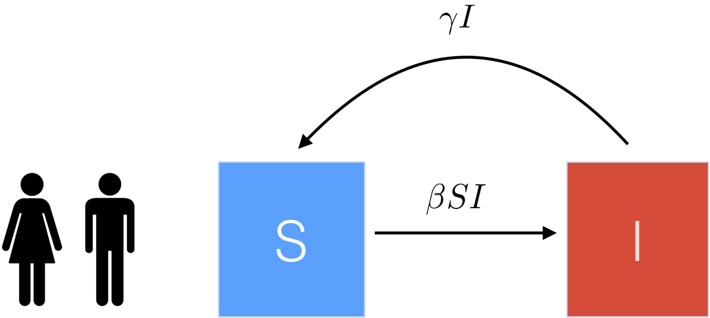

Figure: The SIS model.

Introduction to the SIS Model

1. The SIS model can be written as a differential equations system form.

$$\begin{cases} \frac{dS}{dt} = -\beta SI + \gamma I, \\ \frac{dI}{dt} = \beta SI - \gamma I. \end{cases}$$

2. There are several kinds of extensions to the SIS model, for example, the SIR model.
3. The SIS model is the simplest epidemic model that can describe the phase transitions.

Non-Equilibrium Phase Transition

1. The SIS model has the differential equation form

$$\begin{cases} \frac{dS}{dt} = -\beta SI + \gamma I, \\ \frac{dI}{dt} = \beta SI - \gamma I. \end{cases}$$

2. The model is characterized by two parameters β and γ.
3. The non-equilibrium phase transition in SIS models is described by the relation

$$\begin{cases} \textbf{Absorbing Phase: } \frac{\beta}{\gamma} \leq 1, \\ \textbf{Endemic Phase: } \frac{\beta}{\gamma} > 1. \end{cases}$$

4. Simulation:
`https://kikiyenhaoyang.github.io/kikiyen/Web/TN.html`.

From Ising model to the SIS mode

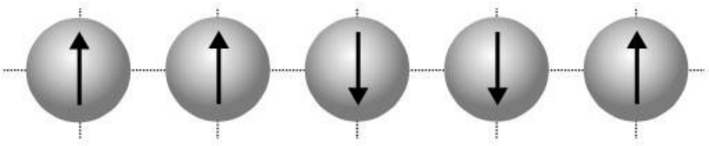

Figure: One-dimensional Ising model.

From Ising model to the SIS mode

1. Each lattice in the model having the quantum state $|s_i\rangle$ lives in the Hilbert space and therefore can be written as the linear combination of $|0\rangle$ and $|1\rangle$.
2. **Composite Systems Postulate**
   The state space of a composite quantum system is the tensor product of the state spaces of its components. That is, the complete Hilbert space of a composite system is

$$\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i.$$

From Ising model to the SIS mode

1. The basis for the complete Hilbert space is

$$\{\bigotimes_{j=1}^{N} |i_j\rangle\}_{i_j \in \{0,1\}}.$$

2. The quantum state of the complete Ising model can be written as

$$|\Psi\rangle = \Psi(s_1, \cdots, s_N) \bigotimes_{i=1}^{N} |s_i\rangle .$$

From Ising Model to the SIS Model

1. The Hibert space has dimension $2^N = \mathcal{O}(2^N)$, which grows rapidly.
2. With the TN, we can reduce it to $\mathcal{O}(N)$, which is much more efficient.
3. The SIS model is also a kind of binary model. Thus we can write it in tensor product formulation as well.

The Tensor Product Formulation

1. **Measurement Postulate**
   Upon measurement, the outcome of an observable is one of its eigenvalues, and the probability of obtaining a particular outcome is given by the square of the absolute value of the projection of the state vector onto the corresponding eigenvector. That is,

$$\sum_i C_i \left|\Psi_i\right\rangle \xrightarrow{\text{Measurement}} \left|\Psi_i\right\rangle, \text{ with probability } |C_i|^2.$$

The Tensor Product Formulation

1. In the lattice form SIS model. A lattice is in state $|0\rangle = |S\rangle$ or $|1\rangle = |I\rangle$ with probability $P_0(t)$ and $P_1(t)$ respectively. That is,

$$|P(t)\rangle = P_0(t)\,|0\rangle + P_1(t)\,|1\rangle\,,$$

which is analog to

$$|\Psi\rangle = \Psi_0\,|0\rangle + \Psi_1\,|1\rangle\,.$$

2. The state space of the SIS model is

$$\bigotimes_{i=1}^{N} \mathcal{H}_i, \text{ where } \mathcal{H} = \text{Span}\{|0\rangle\,, |1\rangle\}.$$

The Tensor Product Formulation

1. Since the SIS model is a kind of Markov chain, we can consider the infinitesimal Markov generator $\hat{W}$ so that

$$\partial_t \left| P(t) \right\rangle = \hat{W} \left| P(t) \right\rangle ,$$

which is Schrödinger-equation-like.

2. **Time Evolution Postulate**
The time evolution of a quantum system is governed by the Schrödinger equation

$$-i\partial_t \left| \Psi \right\rangle = \hat{H} \left| \Psi \right\rangle ,$$

which describes how the state vector changes over time.

The Tensor Product Formulation

1. The infinitesimal Markov generator $\hat{W}$ for the SIS model is

$$\hat{W} = \beta \sum_{i=1}^{N-1} (\hat{n}_i \hat{\omega_{i+1}}^{0 \longrightarrow 1} + \hat{\omega}_i{}^{0 \longrightarrow 1} \hat{n_{i+1}}) + \gamma \sum_{i=1}^{N} \hat{\omega}_i{}^{1 \longrightarrow 0} + \hat{W}_{driv}(\alpha),$$

   where

$$\hat{n}_i = \left|1\right\rangle_i \left\langle 1\right| , \; \hat{\omega}_i{}^{0 \longrightarrow 1} = \left|1\right\rangle_i \left\langle 0\right| - \left|0\right\rangle_i \left\langle 0\right| , \; \hat{\omega}_i{}^{0 \longrightarrow 1} = \left|0\right\rangle_i \left\langle 1\right| - \left|1\right\rangle_i \left\langle 1\right| .$$

2. The generator is *analog* to the quantum Hamiltonian.

The Tensor Product Formulation

1. **Quantum State Postulate**
   A quantum system is fully described by its state vector, usually denoted by $|\psi\rangle$ in Dirac notation. This state vector resides in a complex vector space known as a $\mathbb{C}-$Hilbert space.

The Tensor Product Formulation

Definition

Vector Space

*Let F be a field. A F-**vector space** is set V equipped with two operators*

$$+ : V \times V \to V,$$

$$\cdot : F \times V \to V$$

satisfies the following conditions

- $(V, +)$ *is an abelian group,*
- $a \cdot (b \cdot v) = (ab) \cdot v$ *for all* $a,\ b \in F$ *and* $v \in V$,
- $(a + b) \cdot v = a \cdot v + b \cdot v$ *for all* $a,\ b \in F$ *and* $v \in V$,
- $a(u + v) = au + av$ *for all* $a \in F$ *and* $u,\ v \in V$,
- $id_F \in F$ *such that* $id_F \cdot v = v$.

The Tensor Product Formulation

1. Though we analog the generator to the quantum Hamiltonian, there are many differences between the generator and the quantum Hamiltonian.
2. The generator is a **REAL** matrix and the the basis lives in a $\mathbb{R}-$Hilbert space.
3. Many good properties are broken in the real Hilbert space.
4. We can not treat it as a quantum system.