Group Meeting #9

Hao-Yang Yen

NTHU

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The Large Deviation Principle

Definition

If a variable A_n parametrized by a number n satisfies the relation

$$\lim_{n \longrightarrow \infty} -\frac{1}{n} \ln P(A_n \in B) = I_B,$$

or equivalently we can write it in little-o notation

$$\ln P(A_n \in B) = -nI_B + o(n),$$

then we say that A_n satisfies the large deviation principle (LD) with rate I_B.

The Large Deviation Principle

Example

Entropy

Consider a non-interaction spin chain with n lattices. The mean value of magnetisation is $m = \langle M \rangle = \frac{1}{n} \sum_i \sigma_i$. *We therefore have the degenerate number of each m*

$$\Gamma(m) = \frac{n!}{[(1-m)n/2]![(1+m)n/2]!}.$$

The Large Deviation Principle

Example

Entropy *(continued)*
By Stirling approximation, we know that as n *is very big*

$$\Gamma(m) \approx e^{nS(m)},$$

where the S *entropy with respect to* m *is defined as*

$$S(m) = -\frac{1-m}{2}\ln\frac{1-m}{2} - \frac{1+m}{2}\ln\frac{1+m}{2} \geq 0.$$

The Large Deviation Principle

Example

Entropy *(continued)*

We can see that

$$\sup_{m\in(-1,1)} \Gamma(m) = e^{n\ln 2},$$

as $m = 0$ *and therefore* $S(m)$ *is maximized. This is called the Laplace approximation or the saddle point approximation.*

The Large Deviation Principle

Definition

Scaled Cumulant Generating Function

Let A_n be a real random variable parameterized by $n \in \mathbb{N}/\{0\}$. The scaled cumulant generating function $\lambda : \mathbb{R} \to \mathbb{R}$ is defined as

$$\lambda(k) = \lim_{n \to \infty} \frac{1}{n} \ln \left\langle e^{nkA_n} \right\rangle .$$

The Large Deviation Principle

Theorem

The Gartner-Ellis Theorem

Let A_n be a real random variable parameterized by $n \in \mathbb{N}/\{0\}$. If the scaled cumulant generating function of A_n λ exists and is differentiable, then A_n satisfies the LD

$$P(A_n = a) \sim e^{-nI(a)},$$

where

$$I = \sup_{k \in \mathbb{R}} (ka - \lambda(k)).$$