Vector Analysis with Differential Forms for Physics

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Contents

Preface

This note is supplementary material for the 2025 Theoretical Mechanics course, where I am serving as a TA. Vectors are the most elementary objects in physics, especially for second-year majors beginning mechanics and electromagnetism, which is why many textbooks open with vector analysis. Yet, despite the existence of a mature and elegant framework in linear algebra, standard physics texts (e.g., Marion & Thornton; Griffiths) often retain older, less formal treatments, likely reflecting when they were written.

Alongside teaching, I am conducting research in mathematical physics, and I have come to appreciate how linear algebra underpins virtually all of theoretical physics, including classical mechanics. Within our department, however, there is no single course that systematically develops this beautiful and powerful theory. That gap motivates me to teach core linear-algebra ideas to students in Theoretical Mechanics.

In this note, I present a modern approach to linear algebra and a little bit of abstract algebra, emphasizing the concepts most useful in physics and illustrating them with physics examples and physically motivated examples. Besides teaching, preparing these materials also allows me to revisit some foundational ideas. I aim to share a more rigorous and contemporary perspective on vectors and linear algebra—one that helps students solve physical problems more effectively and, I hope, inspires some to pursue mathematical research in the future.

Chapter 1

Motivation and Background

1.1 Vector Calculus (Physics Viewpoint)

1.1.1 Gradient, divergence, curl

1.1.2 Gauss theorem, Stokes theorem

1.1.3 Limitations of the "vector calculus" notation in curvilinear coordinates

1.2 Linear Algebra Meets Calculus

Chapter 2

Differential Forms

2.1 Concept of Manifolds (without topology)

In many physical systems, the configuration space is not the whole $\mathbb{R}^3$.

- A particle moving on a sphere: $x^2 + y^2 + z^2 = R^2$.
- A pendulum: described by an angle θ, not by (x, y) in $\mathbb{R}^2$.
- A rigid body: described by rotations (three angles, not three coordinates).

We still want to do calculus — take derivatives, define tangent vectors, and integrate fields — on such spaces. This is why we need the concept of **manifolds**

Definition 2.1: Manifolds (physics version)

A **smooth manifold** of dimension n is a space that, **near every point**, looks like $\mathbb{R}^n$. This means we can choose local coordinates $(x^1, \dots, x^n)$ around each point, so that functions, derivatives, and vectors are defined just as in ordinary multivariable calculus.

Globally, the manifold may curve, twist, or wrap around itself, but locally the mathematics is the same as in $\mathbb{R}^n$.

Example 2.1: Sphere S^2

The surface of a sphere is 2-dimensional: locally we can use coordinates (θ, ϕ). Calculus on S^2 (e.g. gradients, integrals) works the same way as in $\mathbb{R}^2$, using these local coordinates.

Definition 2.2: Manifolds and coordinate charts

A smooth manifold M is locally described by coordinate charts

$$\varphi : U \subset M \to \mathbb{R}^n, \qquad p \mapsto (x^1(p), \ldots, x^n(p)).$$

Each chart provides a local coordinate system $(x^1, \ldots, x^n)$ in which we can represent tangent vectors and forms. However, the geometric object itself — a vector, a form, or a field — does not depend on which chart we use.

Remark 2.1: Charts without topology

In rigorous mathematics, a **chart** is a map from a patch of the manifold to an open subset of $\mathbb{R}^n$. For our purposes, you can think of it as a "coordinate patch" covering part of the surface. Different coordinate patches overlap smoothly, just like different map projections of the Earth.

In physics, a manifold is simply a space where you can use coordinates locally, even if you cannot describe it globally by a single coordinate system.

2.2 Tangent and cotangent spaces

Definition 2.3: Tangent space in $\mathbb{R}^n$

At a point $p \in \mathbb{R}^n$, a **tangent vector** at p is the velocity $\dot{\gamma}(0)$ of some smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ with $\gamma(0) = p$. The set of all tangent vectors at p is a vector space called the **tangent space** at p, denoted $T_p\mathbb{R}^n$.

In standard coordinates $x = (x^1, \dots, x^n)$, any curve $\gamma(t) = (x^1(t), \dots, x^n(t))$ with $\gamma(0) = p$ has velocity

$$\dot{\gamma}(0) = \left(\frac{\mathrm{d}x^1}{\mathrm{d}t}, \dots, \frac{\mathrm{d}x^n}{\mathrm{d}t} \right) \Big|_{t=0}.$$

Thus $T_p\mathbb{R}^n$ can be identified with $\mathbb{R}^n$ as a vector space, but conceptually it is "attached" to the base point p.

Remark 2.2: Coordinate basis and directional derivatives

Let $(x^1, \dots, x^n)$ be coordinates on an open set. At p, the vectors

$$\left(\frac{\partial}{\partial x^1} \right)_p, \ \dots, \ \left(\frac{\partial}{\partial x^n} \right)_p$$

form a basis of $T_p\mathbb{R}^n$, characterized by their action on smooth functions f via directional derivatives:

$$\left(\frac{\partial}{\partial x^i} \right)_p [f] \equiv \frac{\partial f}{\partial x^i}.$$

Any $v \in T_p\mathbb{R}^n$ may be written $v = \sum_i v^i (\partial/\partial x^i)_p$ and acts on f by $v[f] = \sum_i v^i \, (\partial f/\partial x^i)_p$.

Example 2.2: Tangent plane to a surface

Let a surface in $\mathbb{R}^3$ be given as a level set $F(x, y, z) = 0$ with $\nabla F(p) \neq 0$. Then the **tangent plane** at p is

$$T_pS = \{ \, v \in T_p\mathbb{R}^3 \mid \nabla F(p) \cdot v = 0 \, \}.$$

Equivalently, if S is parametrized by $\Phi(u,v)$, then $T_pS = \mathrm{span}\{\frac{\partial\Phi}{\partial u}, \frac{\partial\Phi}{\partial v}\}$ at $p = \Phi(u_0, v_0)$.

Definition 2.4: Cotangent space and 1-forms

The **cotangent space** at p, denoted $T_p^*\mathbb{R}^n$, is the space of all linear maps (also called **covectors** or **1-forms at** p)

$$\omega_p : T_p\mathbb{R}^n \to \mathbb{R}.$$

There is a natural pairing $\langle \omega_p, v\rangle := \omega_p(v)$ between covectors and vectors.

Remark 2.3: Coordinate 1-forms

The differentials $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ at p form the dual basis of $T_p^*\mathbb{R}^n$, defined by $\mathrm{d}x^i\big((\partial/\partial x^j)_p\big) = \delta^i{}_j$. Any 1-form can be written $\omega = \sum_i \omega_i \, \mathrm{d}x^i$.

2.3 Charts and Vectors in Physics

Theorem 2.1: Chart-change (Jacobian) law for vectors and covectors

Let $(x^1, \ldots, x^n)$ and $(\tilde{x}^1, \ldots, \tilde{x}^n)$ be two smooth coordinate systems on an overlap region of M, with $\tilde{x} = \tilde{x}(x)$ a smooth change of variables. At $p \in M$, write the Jacobian and its inverse as

$$J^j{}_i := \frac{\partial \tilde{x}^j}{\partial x^i}, \qquad (K)^i{}_j := \frac{\partial x^i}{\partial \tilde{x}^j} = (J^{-1})^i{}_j.$$

Then:

1. (Vectors are contravariant.) If $v \in T_pM$ has components v^i in x-

coordinates and $\tilde{v}^j$ in $\tilde{x}$-coordinates, then

$$\tilde{v}^j = J^j{}_i \, v^i.$$

2. (Covectors are covariant.) If $\omega \in T_p^*M$ has components ω_i and $\tilde{\omega}_j$ in the two charts, then
$$\tilde{\omega}_j = K^i{}_j \, \omega_i.$$

3. (Pairing is invariant.) The scalar $\omega(v)$ is coordinate-independent:
$$\tilde{\omega}_j \, \tilde{v}^j = \omega_i \, v^i.$$

Proof (chain rule). For a smooth f, the directional derivative is $v[f] = v^i \frac{\partial f}{\partial x^i} = \tilde{v}^j \frac{\partial f}{\partial \tilde{x}^j}$. By the chain rule, $\dfrac{\partial f}{\partial x^i} = \dfrac{\partial \tilde{x}^j}{\partial x^i} \dfrac{\partial f}{\partial \tilde{x}^j} = J^j{}_i \, \dfrac{\partial f}{\partial \tilde{x}^j}$. Comparing coefficients gives $\tilde{v}^j = J^j{}_i v^i$. Define covectors by their action on vectors and require $\omega(v)$ to be invariant in all charts: $\omega_i v^i = \tilde{\omega}_j \tilde{v}^j = \tilde{\omega}_j J^j{}_i v^i$ for all v, so $\omega_i = (J^\top)_i{}^j \tilde{\omega}_j$, i.e. $\tilde{\omega}_j = (J^{-1})^i{}_j \omega_i = K^i{}_j \omega_i$. The pairing invariance follows. □

Example 2.3: Chart change and the meaning of vector components

Consider two coordinate systems on $\mathbb{R}^3$: the Cartesian coordinates (x, y, z) and the spherical coordinates (r, θ, ϕ) related by

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta.$$

A vector at a point p can be written either in the Cartesian basis

$$v = v^x \frac{\partial}{\partial x} + v^y \frac{\partial}{\partial y} + v^z \frac{\partial}{\partial z},$$

or in the spherical basis

$$v = v^r \frac{\partial}{\partial r} + v^\theta \frac{\partial}{\partial \theta} + v^\phi \frac{\partial}{\partial \phi}.$$

Applying the chain rule,

$$\frac{\partial}{\partial r} = \sin\theta\cos\phi\,\frac{\partial}{\partial x} + \sin\theta\sin\phi\,\frac{\partial}{\partial y} + \cos\theta\,\frac{\partial}{\partial z},$$

and similarly for $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\phi}$. Thus, the components of the same geometric vector in the two charts are related by the Jacobian:

$$\begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} v^r \\ v^\theta \\ v^\phi \end{pmatrix}.$$

Although the numerical components change with coordinates, the geometric vector v itself is the same object: a tangent direction at the point p.
Similarly, a covector (1-form)

$$\omega = \omega_x\,\mathrm{d}x + \omega_y\,\mathrm{d}y + \omega_z\,\mathrm{d}z = \omega_r\,\mathrm{d}r + \omega_\theta\,\mathrm{d}\theta + \omega_\phi\,\mathrm{d}\phi$$

transforms with the *inverse* Jacobian, so that the pairing $\omega(v) = \omega_i v^i$ is invariant under the change of chart.
Changing coordinates only reshuffles the components. The physical or geometric quantity itself does not change.

Example 2.4: Cylindrical coordinates: chart change and volume, area forms

Consider Cartesian (x, y, z) and cylindrical (ρ, ϕ, z) related by

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z.$$

Vector components (contravariant). Write the same geometric vector v as

$$v = v^x\frac{\partial}{\partial x} + v^y\frac{\partial}{\partial y} + v^z\frac{\partial}{\partial z} = v^\rho\frac{\partial}{\partial\rho} + v^\phi\frac{\partial}{\partial\phi} + v^z\frac{\partial}{\partial z}.$$

Using the chain rule,
$$\frac{\partial}{\partial\rho} = \cos\phi\,\frac{\partial}{\partial x} + \sin\phi\,\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial\phi} = -\rho\sin\phi\,\frac{\partial}{\partial x} + \rho\cos\phi\,\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial z}.$$
Hence
$$\begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Jacobian } \partial(x,y,z)/\partial(\rho,\phi,z)} \begin{pmatrix} v^\rho \\ v^\phi \\ v^z \end{pmatrix}.$$
Although the *numbers* change with the chart, the vector v itself does not.

Covector (1-form) components (covariant). Let

$$\omega = \omega_x\,\mathrm{d}x + \omega_y\,\mathrm{d}y + \omega_z\,\mathrm{d}z = \omega_\rho\,\mathrm{d}\rho + \omega_\phi\,\mathrm{d}\phi + \omega_z^{(\mathrm{cyl})}\,\mathrm{d}z.$$

First express differentials:

$$\mathrm{d}\rho = \cos\phi\,\mathrm{d}x + \sin\phi\,\mathrm{d}y, \qquad \mathrm{d}\phi = \frac{-\sin\phi}{\rho}\,\mathrm{d}x + \frac{\cos\phi}{\rho}\,\mathrm{d}y, \qquad \mathrm{d}z = \mathrm{d}z.$$
Comparing coefficients gives the inverse–Jacobian law
$$\omega_\rho = \cos\phi\,\omega_x + \sin\phi\,\omega_y, \qquad \omega_\phi = \frac{-\sin\phi}{\rho}\,\omega_x + \frac{\cos\phi}{\rho}\,\omega_y, \qquad \omega_z^{(\mathrm{cyl})} = \omega_z,$$
so the pairing $\omega(v)$ is chart-invariant.

Volume and area forms. The standard volume form transforms by the Jacobian determinant:

$$\mathrm{d}V = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \rho\,\mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}z.$$

Typical oriented area elements:

$$\text{plane } z = \text{const}: \quad \mathrm{d}S = \mathrm{d}x \wedge \mathrm{d}y = \rho\,\mathrm{d}\rho \wedge \mathrm{d}\phi,$$
$$\text{cylinder } \rho = R: \quad \mathrm{d}S = R\,\mathrm{d}\phi \wedge \mathrm{d}z \quad (\text{outward normal}).$$

Physical takeaway. In cylindrical coordinates the ϕ-direction basis $\frac{\partial}{\partial\phi}$ has length scale ρ, so components along ϕ naturally carry a factor ρ. This is a chart effect, not a change of the underlying physical vector or 1-form.

Remark 2.4: Bases transform dually

Coordinate vector fields and 1-forms transform as

$$\left(\frac{\partial}{\partial x^i}\right)_p = J^j{}_i \left(\frac{\partial}{\partial \tilde{x}^j}\right)_p, \qquad \mathrm{d}\tilde{x}^j_p = J^j{}_i \, \mathrm{d}x^i_p, \quad \mathrm{d}x^i_p = K^i{}_j \, \mathrm{d}\tilde{x}^j_p.$$

Thus components transform with J or $K = J^{-1}$ so that the pairing $\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i{}_j$ stays intact.

Corollary 2.1: General tensor chart-change rule

A (k, ℓ)-tensor T has components $T^{i_1 \cdots i_k}{}_{j_1 \cdots j_\ell}$ in x-coords and $\tilde{T}^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell}$ in $\tilde{x}$-coords related by

$$\tilde{T}^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell} = J^{a_1}{}_{i_1} \cdots J^{a_k}{}_{i_k} \, K^{j_1}{}_{b_1} \cdots K^{j_\ell}{}_{b_\ell} \, T^{i_1 \cdots i_k}{}_{j_1 \cdots j_\ell}.$$

Contravariant indices pick up J; covariant indices pick up $K = J^{-1}$.

Proposition 2.1: Top-degree forms and Jacobians

For an n-form Ω in n dimensions,

$$\tilde{\Omega} = \tilde{\Omega}_{1 \cdots n} \, \mathrm{d}\tilde{x}^1 \wedge \cdots \wedge \mathrm{d}\tilde{x}^n = \Omega_{1 \cdots n} \, \det(J) \, \mathrm{d}\tilde{x}^1 \wedge \cdots \wedge \mathrm{d}\tilde{x}^n.$$

In particular, the standard volume form transforms by the determinant:

$$\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \det(K) \, \mathrm{d}\tilde{x}^1 \wedge \cdots \wedge \mathrm{d}\tilde{x}^n = \frac{1}{\det(J)} \, \mathrm{d}\tilde{x}^1 \wedge \cdots \wedge \mathrm{d}\tilde{x}^n.$$

Example 2.5: Spherical coordinates and the volume, area elements

In $\mathbb{R}^3$, let $\tilde{x} = (r, \theta, \phi)$ with $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$. Compute $J^j{}_i = \partial \tilde{x}^j / \partial x^i$ or equivalently $K = \partial x / \partial \tilde{x}$ and get $\det(K) = r^2 \sin\theta$.

Hence, the Euclidean volume form becomes

$$\mathrm{d}V = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = r^2 \sin\theta \,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi.$$

Restricting to the sphere $r = R$ gives the area 2-form

$$\mathrm{d}S_{S^2} = R^2 \sin\theta \,\mathrm{d}\theta \wedge \mathrm{d}\phi,$$

which matches the familiar surface element from vector calculus.

A **passive** change of chart re-expresses the same geometric vector with new components (via J and K). An **active** transformation would move the vector or point itself. In this section, we only used the passive, coordinate-change picture—exactly what ensures that physical laws (equations between tensors) are form-invariant in any coordinates.

In summary,

- Vectors: $\tilde{v} = J\,v$ (contravariant).
- Covectors: $\tilde{\omega} = K^\top \omega$ or $\tilde{\omega}_j = K^i{}_j \omega_i$ (covariant).
- Pairing invariant: $\omega(v)$ unchanged.
- n-forms or volume: pick up det of the Jacobian of the coordinate map.
- Tensors (k, ℓ): each upper index gets a J, each lower index a $K = J^{-1}$.

Definition 2.5: Vectors as coordinate-independent objects

At each point $p \in M$, a tangent vector $v \in T_pM$ acts as a directional derivative:

$$v[f] = \sum_i v^i \frac{\partial f}{\partial x^i}.$$

If we change coordinates from x^i to $\tilde{x}^j = \tilde{x}^j(x)$, the components transform as

$$\tilde{v}^j = \sum_i \frac{\partial \tilde{x}^j}{\partial x^i} v^i.$$

Thus, v is the same geometric entity even though its components change.

In physics, a vector quantity such as velocity or force is often visualized as an arrow in space. Mathematically, it is a coordinate-independent object living in the tangent space T_pM. The components (v^x, v^y, v^z) depend on the chosen coordinates, but the *vector itself* does not.

This explains why physical laws written in vector form (e.g. $\mathbf{F} = m\mathbf{a}$, Maxwell's equations) must hold in all coordinate systems: they are statements about geometric objects, not about their components.

Example 2.6: Change of coordinates and physical vectors

Suppose a particle moves on a sphere of radius R. In Cartesian coordinates (x, y, z) its velocity vector has components $(\dot{x}, \dot{y}, \dot{z})$. In spherical coordinates (θ, ϕ), the same velocity is expressed as

$$v = R\,\dot{\theta}\,\frac{\partial}{\partial x^\theta} + R\sin\theta\,\dot{\phi}\,\frac{\partial}{\partial x^\phi}.$$

The change of components reflects only the change of chart, not a change in the physical velocity.

Remark 2.5: Covectors and coordinate transformations

Similarly, 1-forms transform with the inverse Jacobian:

$$\tilde{\omega}_j = \sum_i \frac{\partial x^i}{\partial \tilde{x}^j}\,\omega_i.$$

This dual behavior ensures that the pairing $\omega(v) = \sum_i \omega_i v^i$ is invariant under coordinate changes.

2.4 Vector fields

Definition 2.6: Vector field (physics-friendly definition)

A **vector field** on a smooth space M assigns to each point $p \in M$ a tangent vector $V(p) \in T_pM$, smoothly depending on p. Equivalently, a vector field is a derivation on smooth functions:

$$V: \; C^\infty(M) \to C^\infty(M), \qquad V[f](p) = V(p)[f],$$

linear over $\mathbb{R}$ and satisfying the Leibniz rule $V[fg] = f\,V[g] + g\,V[f]$.

In local coordinates $(x^1, \ldots, x^n)$ on an open set $U \subseteq M$, a vector field reads

$$V = \sum_{i=1}^{n} V^i(x)\,\frac{\partial}{\partial x^i}, \qquad x \in U,$$

where the component functions V^i are smooth (C^∞). Under a chart change $x \mapsto \tilde{x}(x)$ with Jacobian $J^j{}_i = \partial\tilde{x}^j/\partial x^i$, components transform contravariantly:

$$\tilde{V}^j(\tilde{x}) = J^j{}_i(x)\,V^i(x).$$

Thus, V is a coordinate-independent geometric object, even though its components change.

Remark 2.6

Some physics textbooks use this coordinate-independent geometric property to define what vectors are. But in a more modern viewpoint, this is only a result of a change in the chart.

Example 2.7: Basic examples in $\mathbb{R}^3$

- **Uniform field:** $V = a\,\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial y} + c\,\frac{\partial}{\partial z}$ with constants a, b, c.
- **Rotation about the z-axis:** $V = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}$ (generates planar rotations).
- **Radial field:** $V = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + z\,\frac{\partial}{\partial z}$ (points outward, scales with radius).

Example 2.8: Cylindrical coordinates (ρ, ϕ, z)

With $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, the coordinate vector fields are

$$\frac{\partial}{\partial \rho} = \cos\phi\,\frac{\partial}{\partial x} + \sin\phi\,\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \phi} = -\rho\sin\phi\,\frac{\partial}{\partial x} + \rho\cos\phi\,\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z}.$$

Hence the same vector field may be written

$$V = V^\rho\,\frac{\partial}{\partial \rho} + V^\phi\,\frac{\partial}{\partial \phi} + V^z\,\frac{\partial}{\partial z} = V^x\,\frac{\partial}{\partial x} + V^y\,\frac{\partial}{\partial y} + V^z\,\frac{\partial}{\partial z},$$

with components related by

$$\begin{pmatrix} V^x \\ V^y \\ V^z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^\rho \\ V^\phi \\ V^z \end{pmatrix}.$$

The ϕ-component carries a factor ρ because the basis vector $\partial/\partial\phi$ has length scale ρ.

Remark 2.7: Action on functions and gradient pairing

Given a scalar field $f \in C^\infty(M)$ and a vector field V, the directional derivative along V is the new scalar field $V[f]$. If $\mathrm{d}f$ denotes the 1-form (covector field) differential of f, then

$$V[f] = \langle \mathrm{d}f, V \rangle = \sum_i \frac{\partial f}{\partial x^i} V^i$$

in any coordinates. This pairing is coordinate-invariant.

Definition 2.7: Integral curves and flow (brief)

An **integral curve** of V is a curve $\gamma(t)$ satisfying $\dot{\gamma}(t) = V(\gamma(t))$ with initial condition $\gamma(0) = p$. Locally (under mild conditions) there exists a **flow** Φ_t generated by V such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(p) = V(\Phi_t(p)) \quad \text{and} \quad \Phi_0 = \mathrm{id}.$$

Physically, V may represent a velocity field and Φ_t the motion of tracers.

Example 2.9: Physics meanings

- **Velocity field** $\mathbf{v}(x)$ in fluid flow: the integral curves are particle paths.
- **Force field** $\mathbf{F}(x)$ for a charged particle (with given dynamics): integral curves approximate trajectories in configuration space when reparametrized appropriately.
- **Angular velocity field** for a rigid rotation: $V = \boldsymbol{\omega} \times \mathbf{r}$.

2.5 From vectors to covectors

Physics frequently identifies gradients with *vectors* through the Euclidean dot product, but in differential geometry the primary object is the *differential* of a function.

Definition 2.8: Differential of a function

For a smooth $f : \mathbb{R}^n \to \mathbb{R}$ and $p \in \mathbb{R}^n$, the **differential** $\mathrm{d}f_p \in T_p^*\mathbb{R}^n$ is the linear map

$$\mathrm{d}f_p(v) := v[f] = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}, \qquad v = \sum_i v^i \left(\frac{\partial}{\partial x^i}\right)_p.$$

In coordinates, $\mathrm{d}f = \sum_i \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i$.

Remark 2.8: Gradient and differential

Given an inner product (metric) g on $T_p\mathbb{R}^n$, the **musical isomorphism** $v \mapsto v^\flat$ sends a vector to a covector via $v^\flat(\cdot) = g(v,\cdot)$. The gradient ∇f is the unique vector with $g(\nabla f,\cdot) = \mathrm{d}f(\cdot)$, i.e., $\mathrm{d}f = (\nabla f)^\flat$. In Euclidean space with the standard dot product, this reduces to the familiar identification $\mathrm{d}f = \nabla f \cdot \mathrm{d}x$.

Example 2.10: Linear function

Let $f(x) = a_1x^1 + \cdots + a_nx^n$. Then $\mathrm{d}f = \sum_i a_i\,\mathrm{d}x^i$. For any $v = \sum_i v^i(\partial/\partial x^i)$, we have $\mathrm{d}f(v) = \sum_i a_iv^i$.

Example 2.11

Show that $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$ and $\mathrm{d}(\phi \circ f) = \phi'(f)\,\mathrm{d}f$ for smooth $f, g : \mathbb{R}^n \to \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$.

2.5.1 Application: work form $F \cdot \mathrm{d}x$

In vector calculus, the **work** along a curve $\gamma : [a,b] \to \mathbb{R}^n$ under a force field $F : \mathbb{R}^n \to \mathbb{R}^n$ is

$$W = \int_\gamma F \cdot \mathrm{d}x := \int_a^b F(\gamma(t)) \cdot \dot\gamma(t)\,\mathrm{d}t.$$

Define the associated 1-form (the **work form**)

$$\omega := F \cdot \mathrm{d}x = \sum_{i=1}^{n} F_i(x)\,\mathrm{d}x^i.$$

Then the line integral is simply the integral of the 1-form ω along γ

$$W = \int_\gamma \omega.$$

Example 2.12: Circulation around a circle in $\mathbb{R}^2$

Let $F(x,y) = (-y, x)$ on $\mathbb{R}^2$. Then

$$\omega = F \cdot \mathrm{d}x = -y\,\mathrm{d}x + x\,\mathrm{d}y.$$

Take the unit circle $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Compute

$$\int_\gamma \omega = \int_0^{2\pi} \Big(-\sin t\,(-\sin t) + \cos t\,(\cos t) \Big) \mathrm{d}t \tag{2.1}$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,\mathrm{d}t = 2\pi. \tag{2.2}$$

Remark 2.9: Closed and exact (preview)

Write $\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y$. In $\mathbb{R}^2$, the exterior derivative of a 1-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$ is

$$\mathrm{d}\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y.$$

Here $\mathrm{d}\omega = (1 - (-1))\,\mathrm{d}x \wedge \mathrm{d}y = 2\,\mathrm{d}x \wedge \mathrm{d}y \neq 0$, so ω is *not* closed and hence not exact; accordingly, its integral around a loop can be nonzero. This foreshadows Stokes' theorem.

Example 2.13: Conservative field

Let $F = \nabla\phi$ for a smooth potential $\phi : \mathbb{R}^n \to \mathbb{R}$. Then $\omega = F \cdot \mathrm{d}x = \mathrm{d}\phi$ is *exact*. For any curve γ,

$$\int_\gamma \omega = \int_\gamma \mathrm{d}\phi = \phi(\gamma(b)) - \phi(\gamma(a)),$$

so the work depends only on endpoints.

Example 2.14: Work in 3D and curl

Let $F(x, y, z) = \big(y, -x, 0\big)$ on $\mathbb{R}^3$ and $\omega = F \cdot \mathrm{d}x$.

(a) Compute $\boldsymbol{\nabla} \times F$ and verify that $\mathrm{d}\omega$ corresponds to $\boldsymbol{\nabla} \times F$ via the usual identification between 2-forms and pseudovectors in $\mathbb{R}^3$.

(b) Evaluate $\displaystyle\oint_\gamma \omega$ around the unit circle in the xy-plane and relate it to $\iint_S \mathrm{d}\omega$ over the unit disk using Stokes' theorem.

2.6 Higher-degree forms

Definition 2.9: k-forms

A **k-form** on $\mathbb{R}^n$ is a smooth, multilinear, antisymmetric map

$$\omega_p : (T_p\mathbb{R}^n)^k \longrightarrow \mathbb{R}$$

that assigns a real number to each ordered k-tuple of tangent vectors at p. In coordinates, any k-form can be expressed as a linear combination of wedge products of the coordinate 1-forms:

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \cdots i_k}(x)\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

where the coefficients $\omega_{i_1\cdots i_k}(x)$ are smooth functions.

Remark 2.10: Wedge product

Given a p-form α and a q-form β, their **wedge product** $\alpha\wedge\beta$ is a $(p+q)$-form defined by

$$(\alpha\wedge\beta)(v_1,\ldots,v_{p+q}) = \frac{1}{p!\,q!}\sum_{\sigma\in S_{p+q}} \operatorname{sgn}(\sigma)\,\alpha(v_{\sigma(1)},\ldots,v_{\sigma(p)})\,\beta(v_{\sigma(p+1)},\ldots,v_{\sigma(p+q)}).$$

It is bilinear and *graded-antisymmetric*:

$$\alpha\wedge\beta = (-1)^{pq}\,\beta\wedge\alpha.$$

Example 2.15: Area 2-form in $\mathbb{R}^3$

n $\mathbb{R}^3$, the basic 2-forms are $\mathrm{d}x\wedge\mathrm{d}y$, $\mathrm{d}y\wedge\mathrm{d}z$, and $\mathrm{d}z\wedge\mathrm{d}x$. A general 2-form is

$$\omega = P\,\mathrm{d}y\wedge\mathrm{d}z + Q\,\mathrm{d}z\wedge\mathrm{d}x + R\,\mathrm{d}x\wedge\mathrm{d}y,$$

where (P,Q,R) are smooth functions. The correspondence with a pseudovector $\mathbf{F}=(P,Q,R)$ is what allows us to identify $\mathrm{d}\omega$ with $\nabla\cdot\mathbf{F}$ later.

Example 2.16: Volume form

In $\mathbb{R}^3$, the standard **volume form** is

$$\mathrm{d}V = \mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z.$$

For three vectors $v_1,v_2,v_3\in T_p\mathbb{R}^3$,

$$\mathrm{d}V(v_1,v_2,v_3) = \det[v_1\ v_2\ v_3],$$

which gives the oriented volume of the parallelepiped they span. In $\mathbb{R}^n$, the corresponding form is $\mathrm{d}V = \mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n$.

Remark 2.11: Interpretation

- 1-forms $\longleftrightarrow$ line elements, integrated along curves.
- 2-forms $\longleftrightarrow$ oriented area elements, integrated over surfaces.
- 3-forms $\longleftrightarrow$ volume elements, integrated over regions.

The exterior derivative d connects them:

$$\text{0-forms (functions)} \xrightarrow{\mathrm{d}} \text{1-forms} \xrightarrow{\mathrm{d}} \text{2-forms} \xrightarrow{\mathrm{d}} \text{3-forms}, \qquad \mathrm{d}^2 = 0.$$

This hierarchy underlies the integral theorems of vector calculus.

2.7 Exterior Derivative and Vector Calculus

Definition 2.10: Exterior derivative

The **exterior derivative** d is an operator that maps k-forms to $(k+1)$-forms,

$$\mathrm{d} : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n),$$

characterized by the following properties:

1. For a function (0-form) f, $\mathrm{d}f = \sum_i \dfrac{\partial f}{\partial x^i}\,\mathrm{d}x^i$.
2. For general ω and η,
$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k\,\omega \wedge \mathrm{d}\eta,$$
if ω is a k-form.
3. $\mathrm{d}(\mathrm{d}\omega) = 0$ for all ω (that is, $\mathrm{d}^2 = 0$).

Example 2.17: Computation in coordinates

Let $\omega = \sum_i f_i \, \mathrm{d}x^i$ be a 1-form. Then

$$\mathrm{d}\omega = \sum_{i<j} \left(\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} \right) \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

Similarly, for a 2-form $\omega = \sum_{i<j} f_{ij} \, \mathrm{d}x^i \wedge \mathrm{d}x^j$,

$$\mathrm{d}\omega = \sum_{i<j<k} \left(\frac{\partial f_{jk}}{\partial x^i} + \frac{\partial f_{ki}}{\partial x^j} + \frac{\partial f_{ij}}{\partial x^k} \right) \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k.$$

One of the main advantages of differential forms is that the familiar vector calculus operators $\nabla, \quad \nabla\times, \quad \nabla\cdot$ can all be unified through the differential forms. This is the theoretical foundation of the modern viewpoint of vector analysis. We first take a glance at these three operators in differential forms.

- **Gradient as a 1-form.** Let f be a scalar function (a 0-form). Its exterior derivative is

  $$\mathrm{d}f = \frac{\partial f}{\partial x} \mathrm{d}x + \frac{\partial f}{\partial y} \mathrm{d}y + \frac{\partial f}{\partial z} \mathrm{d}z.$$

  This is exactly the gradient of f, but expressed as a covector (1-form).

- **Curl as the derivative of a 1-form.** Let $A = A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z$ be a 1-form. Then

  $$\begin{aligned} \mathrm{d}A = &\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathrm{d}y \wedge \mathrm{d}z \\ &+ \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathrm{d}z \wedge \mathrm{d}x \\ &+ \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

  This 2-form encodes the curl of $\mathbf{A}$. Under the identification between 2-forms and pseudovectors in $\mathbb{R}^3$, this is precisely $\nabla \times \mathbf{A}$.

- **Divergence as the derivative of a 2-form.** Let $B = B_x\,\mathrm{d}y \wedge \mathrm{d}z + B_y\,\mathrm{d}z \wedge \mathrm{d}x + B_z\,\mathrm{d}x \wedge \mathrm{d}y$ be a 2-form. Then

$$\mathrm{d}B = \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

  This is a 3-form representing the divergence $\nabla \cdot \mathbf{B}$.

Actually, we can identify these three differential form through some vector identities that we are familiar with:

$$\text{Gradient: } \int_{\mathbf{a}}^{\mathbf{b}} \mathrm{d}f = \int_{\mathbf{a}}^{\mathbf{b}} \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}),$$

$$\text{Curl: } \oint_{\partial S} \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \iint_S \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \iint_S \mathrm{d}A,$$

$$\text{Divergence: } \oiint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \iiint_V \nabla \cdot \mathbf{B} \mathrm{d}V = \iiint_V \mathrm{d}B.$$

Remark 2.12: Geometric meaning

The exterior derivative measures the "infinitesimal circulation" or "flux density" of a form. It generalizes the gradient, curl, and divergence:

$$\begin{array}{lccl} \text{0-form:} & f & \xrightarrow{\mathrm{d}} & \nabla f \cdot \mathrm{d}\mathbf{r}, \\ \text{1-form:} & \mathbf{A} \cdot \mathrm{d}\mathbf{r} & \xrightarrow{\mathrm{d}} & (\nabla \times \mathbf{A}) \cdot (\mathrm{d}\mathbf{S}), \\ \text{2-form:} & \mathbf{B} \cdot (\mathrm{d}\mathbf{S}) & \xrightarrow{\mathrm{d}} & (\nabla \cdot \mathbf{B})\,\mathrm{d}V. \end{array}$$

Here $\mathrm{d}S$ represents the oriented surface element (a 2-form), and $\mathrm{d}V$ the volume form (a 3-form).

Example 2.18: Recovering familiar vector identities in $\mathbb{R}^3$

- If $\omega = \mathrm{d}f$ (a 1-form), then $\mathrm{d}\omega = \mathrm{d}(\mathrm{d}f) = 0$ corresponds to $\boldsymbol{\nabla} \times (\nabla f) = 0$.
- If $\omega = F \cdot \mathrm{d}x$ (a 1-form), then $\mathrm{d}\omega$ corresponds to $\boldsymbol{\nabla} \times F$.

- If $\omega = F \cdot (\mathrm{d}S)$ (a 2-form), then $\mathrm{d}\omega$ corresponds to $\boldsymbol{\nabla} \cdot F$.

Thus, the identity $\mathrm{d}^2 = 0$ encodes the classical vector calculus results:

$$\nabla \times (\nabla f) = 0, \qquad \nabla \cdot (\nabla \times F) = 0.$$

Definition 2.11: Closed and exact forms

A form ω is called **closed** if $\mathrm{d}\omega = 0$, and **exact** if there exists another form η such that $\omega = \mathrm{d}\eta$. Since $\mathrm{d}^2 = 0$, every exact form is automatically closed, but not every closed form is exact. This subtlety lies at the heart of topology (de Rham cohomology) and thermodynamics in physics.

Theorem 2.2: Exterior derivative and vector calculus in $\mathbb{R}^3$

Let $U \subset \mathbb{R}^3$ be an open set. Then the sequence of exterior derivatives

$$\Omega^0(U) \xrightarrow{\ \mathrm{d}\ } \Omega^1(U) \xrightarrow{\ \mathrm{d}\ } \Omega^2(U) \xrightarrow{\ \mathrm{d}\ } \Omega^3(U)$$

corresponds to the sequence of vector calculus operators

$$C^\infty(U) \xrightarrow{\ \nabla\ } \mathfrak{X}(U) \xrightarrow{\ \nabla\times\ } \mathfrak{X}(U) \xrightarrow{\ \nabla\cdot\ } C^\infty(U),$$

where

- $\Omega^k(U)$ is the space of smooth differential k-forms on U,
- $C^\infty(U)$ is the space of smooth functions on U,
- $\mathfrak{X}(U)$ is the space of smooth vector fields on U.

More explicitly, we write

$$\begin{aligned}
&\text{0-form } f \quad \xrightarrow{\ \mathrm{d}\ } \quad \mathrm{d}f \ \longleftrightarrow\ \nabla f,\\
&\text{1-form } A \quad \xrightarrow{\ \mathrm{d}\ } \quad \mathrm{d}A \ \longleftrightarrow\ \nabla \times \mathbf{A},\\
&\text{2-form } B \quad \xrightarrow{\ \mathrm{d}\ } \quad \mathrm{d}B \ \longleftrightarrow\ \nabla \cdot \mathbf{B}.
\end{aligned}$$

Corollary 2.2

The property $d^2 = 0$ encodes familiar vector calculus identities:

$$\nabla \times (\nabla f) = 0, \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

Remark 2.13: Geometric meaning of k-forms

A differential k-form assigns an **oriented density** to a k-dimensional object:

- 0-form: scalar field (value at each point).
- 1-form: oriented line density (acts on tangent vectors).
- 2-form: oriented surface density (acts on pairs of tangent vectors).
- 3-form: oriented volume density.

Each form type naturally integrates over an object of matching dimension.

Example 2.19: 1-forms as line densities

A 1-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$ in $\mathbb{R}^2$ assigns to a vector $v = (v_x, v_y)$ the number $\omega(v) = P\,v_x + Q\,v_y$. This represents the infinitesimal work done by a field $F = (P, Q)$ on a displacement $\mathrm{d}x = v\,\mathrm{d}t$. Thus, integrating ω along a curve measures total work.

Example 2.20: 2-forms as flux densities

A 2-form $\omega = R\,\mathrm{d}y \wedge \mathrm{d}z + S\,\mathrm{d}z \wedge \mathrm{d}x + T\,\mathrm{d}x \wedge \mathrm{d}y$ in $\mathbb{R}^3$ naturally corresponds to a vector field $\mathbf{F} = (R, S, T)$. If v_1, v_2 span a surface patch, then

$$\omega(v_1, v_2) = \mathbf{F} \cdot (v_1 \times v_2)$$

is the oriented flux through that infinitesimal patch. Hence $\displaystyle\int_S \omega$ gives the total flux of $\mathbf{F}$ through S.

Example 2.21: 3-forms as volume densities

3-form such as $\omega = f(x, y, z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ represents a scalar density that integrates to a volume or total quantity:

$$\int_V f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

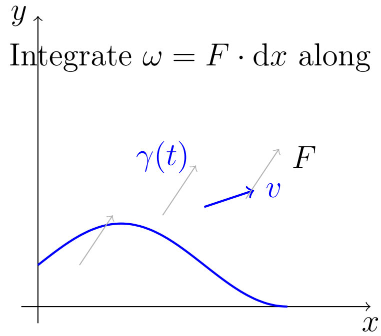
For instance, if f is a mass density, this gives the total mass.

Remark 2.14: The operator d as "boundary detector"

The exterior derivative d measures how a k-form changes across space, capturing oriented "flow through the boundary."

- d of a 0-form (a scalar field) gives its gradient.
- d of a 1-form measures rotational tendency (curl).
- d of a 2-form measures expansion (divergence).

Thus, d transforms local infinitesimal data into the *boundary behavior* that Stokes' theorem then integrates globally.



Thus, the operators gradient, curl, and divergence are unified into a single operation: the exterior derivative d.

Remark 2.15: Unified picture

The hierarchy of forms and d can be summarized as:

$$\begin{array}{ccccccc} f & \xrightarrow{\mathrm{d}} & \nabla f \cdot \mathrm{d}x & \xrightarrow{\mathrm{d}} & (\nabla \times \mathbf{A}) \cdot (\mathrm{d}S) & \xrightarrow{\mathrm{d}} & (\nabla \cdot \mathbf{B})\,\mathrm{d}V \\ \text{0-form} & & \text{1-form} & & \text{2-form} & & \text{3-form} \end{array}$$

and the corresponding integral theorems:

$$\begin{aligned} &\text{Line:} && \int_a^b \mathrm{d}f = f(b) - f(a), && \text{(Fundamental theorem)} \\ &\text{Surface:} && \oint_{\partial S} \omega = \iint_S \mathrm{d}\omega, && \text{(Stokes' theorem)} \\ &\text{Volume:} && \iint_{\partial V} \omega = \iiint_V \mathrm{d}\omega, && \text{(Divergence theorem).} \end{aligned}$$

All are manifestations of the single geometric identity

$$\int_{\partial M} \omega = \int_M \mathrm{d}\omega.$$

Chapter
3

Integration of Differential Forms

After discussing how useful the differential forms are in vector analysis, we move to another important application of differential forms— integration.

3.1 Orientation and integration domains

In physics, integrals of vector fields often measure **flow** or **circulation**. For instance, the flux of a magnetic field through a loop or the total current through a surface both depend not only on the magnitude of the field but also on the chosen **direction** of measurement. To define such quantities consistently, we must specify what "positive direction" means on a curve, surface, or volume. This leads naturally to the notion of **orientation**.

Definition 3.1: Orientation

An **orientation** on a k-dimensional manifold M is a consistent choice of which ordered bases of T_pM are "positive" at every point. For example:

- In $\mathbb{R}$, the direction of increasing x defines the positive orientation.
- In $\mathbb{R}^2$, the orientation of $(\partial/\partial x, \partial/\partial y)$ defines the standard counterclock-

wise sense.

- In $\mathbb{R}^3$, $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ gives the standard right-handed orientation.

Remark 3.1: Why orientation matters

Changing orientation multiplies all integrals of top-degree forms by -1. A k-form can be integrated over a k-dimensional oriented manifold only when the orientation of the domain and of the form agree.

3.2 Integration via pullback

Having established the role of orientation, the next step is to formalize how to compute these oriented integrals. Physically, this corresponds to expressing a curved wire, surface, or volume using local coordinates — just as one parametrizes a membrane by (u, v) to compute its flux. Mathematically, this change of variables is encoded by the **pullback** of differential forms.

Definition 3.2: Pullback of a form

Let $\Phi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a smooth map and let ω be a k-form on $\mathbb{R}^n$. The **pullback** $\Phi^*\omega$ is the k-form on U defined by

$$(\Phi^*\omega)_u(v_1, \ldots, v_k) = \omega_{\Phi(u)}\big((D\Phi)_u(v_1), \ldots, (D\Phi)_u(v_k)\big).$$

In coordinates, if $\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k}(x)\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, then

$$\Phi^*\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k}(\Phi(u))\, \mathrm{d}\big(\Phi^{i_1}\big) \wedge \cdots \wedge \mathrm{d}\big(\Phi^{i_k}\big)\,.$$

Definition 3.3: Integral of a k-form

Let M be an oriented k-dimensional manifold parametrized by $\Phi : U \subset \mathbb{R}^k \to M$. If ω is a k-form on M, its integral over M is defined by

$$\int_M \omega = \int_U \Phi^* \omega.$$

Example 3.1: Line integral of a 1-form

Let $\omega = F \cdot \mathrm{d}x$ be a 1-form on $\mathbb{R}^n$, and let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve. Then

$$\int_\gamma \omega = \int_a^b \gamma^* \omega = \int_a^b F(\gamma(t)) \cdot \dot{\gamma}(t)\, \mathrm{d}t,$$

which is the familiar line integral from vector calculus.

Example 3.2: Surface integral of a 2-form

Let $S \subset \mathbb{R}^3$ be a surface parametrized by $\Phi(u, v)$ and let

$$\omega = P\, \mathrm{d}y \wedge \mathrm{d}z + Q\, \mathrm{d}z \wedge \mathrm{d}x + R\, \mathrm{d}x \wedge \mathrm{d}y.$$

Then

$$\Phi^* \omega = (P, Q, R) \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right) \mathrm{d}u \wedge \mathrm{d}v,$$

so that

$$\int_S \omega = \iint_U (P, Q, R) \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right) \mathrm{d}u\, \mathrm{d}v.$$

This is exactly the oriented flux of the vector field (P, Q, R) through S.

Example 3.3: Volume integral of a 3-form

For a 3-form $\omega = f(x, y, z)\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ and a region $V \subset \mathbb{R}^3$,

$$\int_V \omega = \iiint_V f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

Remark 3.2: Orientation reversal

If Φ reverses orientation (for instance, swapping two coordinates), then $\Phi^*\omega$ picks up a negative sign, hence

$$\int_M \omega = -\int_M (-\omega).$$

This ensures that Stokes' theorem holds consistently under orientation change.

These examples show that integrating forms reproduces all familiar integrals in vector calculus. But the real power of differential forms emerges when we observe that all these separate theorems — for work, flux, and divergence — share one common geometric structure. This leads to the generalized **Stokes' theorem**, which unifies them under a single principle of conservation.

3.3 Stokes' theorem revisited

Theorem 3.1: General Stokes theorem

et M be an oriented k-dimensional manifold with boundary ∂M, and ω a $(k-1)$-form on M. Then

$$\int_{\partial M} \omega = \int_M \mathrm{d}\omega.$$

Example 3.4: Unifying all classical theorems

- For $k = 1$ (intervals in $\mathbb{R}$): Fundamental theorem of calculus.
- For $k = 2$ (surfaces in $\mathbb{R}^3$): Stokes' theorem for curl.
- For $k = 3$ (volumes in $\mathbb{R}^3$): Divergence theorem.

All are instances of $\int_{\partial M} \omega = \int_M \mathrm{d}\omega$.

Example 3.5: Curl and flux

Let $\omega = F \cdot \mathrm{d}x$ with $F(x, y, z) = (y, -x, 0)$ on $\mathbb{R}^3$. Then

$$\mathrm{d}\omega = (\boldsymbol{\nabla} \times F) \cdot (\mathrm{d}S) = (0, 0, -2) \cdot (\mathrm{d}S) = -2\, \mathrm{d}x \wedge \mathrm{d}y.$$

Consider M to be the unit disk $x^2 + y^2 \leq 1$ in the xy-plane. Then ∂M is the unit circle, oriented counterclockwise. By Stokes' theorem:

$$\oint_{\partial M} \omega = \int_M \mathrm{d}\omega = \iint_{x^2+y^2\leq 1} (-2)\, \mathrm{d}x\, \mathrm{d}y = -2\pi.$$

If the boundary orientation were reversed (clockwise), the result would be $+2\pi$.

Remark 3.3: Geometric unification

Differential forms and the exterior derivative provide a single, coordinate-free language for all integral theorems of vector calculus:

- line, surface, and volume integrals are just integrations of forms
- gradient, curl, and divergence are all special cases of exterior derivatives.

This unification is one of the key insights that make differential geometry so powerful in physics.

Remark 3.4: Physical interpretation of Stokes' theorem

In every domain of physics, Stokes' theorem expresses a conservation law:

"flux through the boundary" = "accumulation inside".

For instance:

- Faraday's law: $\displaystyle\oint_{\partial S} E \cdot \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_S B \cdot \mathrm{d}S$, linking electric circulation to changing magnetic flux;

- Gauss's law: $\iint_{\partial V} E \cdot \mathrm{d}S = 4\pi Q_{\text{inside}}$;
- Fluid continuity: $\displaystyle\int_{\partial V} \rho v \cdot \mathrm{d}S = \int_V \boldsymbol{\nabla} \cdot (\rho v)\, \mathrm{d}V$.

Each of these is simply an instance of $\int_{\partial M} \omega = \int_M \mathrm{d}\omega$, showing that differential forms capture the geometric heart of physical laws.

Chapter 4

Applications in Physics

4.1 Electromagnetism in differential forms

Remark 4.1: Motivation

Maxwell's equations take their most elegant and compact form when written in the language of differential forms. The entire set of four equations in vacuum can be expressed as two differential-form equations:

$$\mathrm{d}F = 0, \quad \mathrm{d} \star F = J.$$

In spacetime with coordinates (t, x, y, z), define the **field strength 2-form**

$$\begin{aligned} F = & E_x \, \mathrm{d}x \wedge \mathrm{d}t + E_y \, \mathrm{d}y \wedge \mathrm{d}t + E_z \, \mathrm{d}z \wedge \mathrm{d}t \\ & + B_x \, \mathrm{d}y \wedge \mathrm{d}z + B_y \, \mathrm{d}z \wedge \mathrm{d}x + B_z \, \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

The electric field components appear with $\mathrm{d}t$, and the magnetic field components appear as spatial area elements. This unifies $\mathbf{E}$ and $\mathbf{B}$ into one geometric object.

Applying $\mathrm{d}F = 0$ gives the two homogeneous Maxwell equations:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0. \end{cases}$$

The first expresses the nonexistence of magnetic monopoles; the second expresses Faraday's law of induction.

The **dual form** $\star F$ (using the Hodge star in Minkowski space) is a 2-form that contains the fields $(\mathbf{B}, \mathbf{E})$ in reversed roles. The second Maxwell equation,

$$\mathrm{d} \star F = J,$$

expresses both Gauss's and Ampère–Maxwell laws:

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho, \\ \nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J}. \end{cases}$$

Here J is the **current 3-form**,

$$\begin{aligned} J = & \rho \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - J_x \, \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t \\ & - J_y \, \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}t - J_z \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t. \end{aligned}$$

Remark 4.2: Charge conservation

Applying d to both sides of $\mathrm{d} \star F = J$ gives $\mathrm{d}J = 0$ (because $\mathrm{d}^2 = 0$), which is precisely the local continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

Thus charge conservation is a geometric identity rather than a separate law.

Remark 4.3: Lorentz invariance and compactness

In differential-form language, Maxwell's equations are manifestly Lorentz invariant and coordinate-free. The electromagnetic field is encoded in a single 2-form F, and its dynamics follow from $\mathrm{d}F = 0$ and $\mathrm{d} \star F = J$—a dramatic simplification from four vector equations.

4.2 Fluid flow and vorticity forms

Definition 4.1: Velocity field and vorticity 2-form

Let $\mathbf{v}(x)$ be a smooth velocity field in $\mathbb{R}^3$. The associated 1-form is $v^\flat = v_x\,\mathrm{d}x + v_y\,\mathrm{d}y + v_z\,\mathrm{d}z$. Its exterior derivative,

$$\omega = \mathrm{d}v^\flat = (\nabla \times \mathbf{v})_x\,\mathrm{d}y \wedge \mathrm{d}z + (\nabla \times \mathbf{v})_y\,\mathrm{d}z \wedge \mathrm{d}x + (\nabla \times \mathbf{v})_z\,\mathrm{d}x \wedge \mathrm{d}y,$$

is the **vorticity 2-form**.

This form measures the infinitesimal rotation of the fluid. Integrating ω over a surface gives the total circulation through that surface.

Example 4.1: Kelvin's circulation theorem

n an ideal incompressible fluid, if the flow is governed by an exact form $\omega = \mathrm{d}v^\flat$ with $\mathrm{d}\omega = 0$, then the circulation

$$\oint_{\partial S} v^\flat = \iint_S \omega$$

is conserved as the surface moves with the fluid.

Remark 4.4: Analogy to electromagnetism

The mathematical structure of fluid vorticity parallels that of the magnetic field:

Electromagnetism	**Fluid flow**
1-form: A	1-form: $v^\flat$
2-form: $F = \mathrm{d}A$	2-form: $\omega = \mathrm{d}v^\flat$
Flux conservation: $\mathrm{d}F = 0$	Vorticity conservation: $\mathrm{d}\omega = 0$

This correspondence helps physicists see field theory and fluid dynamics as instances of the same geometric framework.

4.3 Conservation laws as closed forms

Definition 4.2: Conserved current

A **current k-form** J satisfies $\mathrm{d}J = 0$. If M is a $(k+1)$-dimensional region with boundary ∂M, Stokes' theorem implies

$$\int_{\partial M} J = \int_M \mathrm{d}J = 0.$$

Thus, the total flux of J through the boundary vanishes— the geometric expression of a conservation law.

Example 4.2: Charge conservation revisited

In electrodynamics, J is the charge–current 3-form. Because $\mathrm{d}J = 0$, the total charge in any region changes only by the amount of current crossing its boundary:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = -\iint_{\partial V} \mathbf{J}\cdot\mathrm{d}S.$$

Remark 4.5: Unified perspective

Differential forms provide a natural framework for modern physics:

- **Classical mechanics:** Work 1-form $F \cdot \mathrm{d}x$, symplectic 2-form in phase space.
- **Electromagnetism:** Field strength 2-form F and its dual $*F$.
- **Fluid mechanics:** Vorticity 2-form $\mathrm{d}v^\flat$.
- **Relativity and field theory:** Conservation currents as closed forms $\mathrm{d}J = 0$.

Each fundamental conservation law is simply the statement that a certain differential form is closed.